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Periodic orbits in the neighborhood of libration points in certain rotating systems

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PERIODIC ORBITS IN THE NEIGHBORHOOD OF
LIBRATION POINTS IN CERTAIN ROTATING SYSTEMS

by

Victor Wayne Bolie

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

Approved

Signature was redacted for privacy.

In Charge of Major Work

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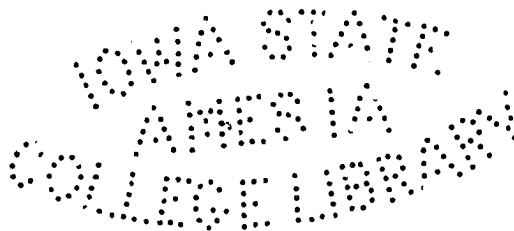
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Dean of Graduate College

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1952



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I. INTRODUCTION

This paper is concerned with the motion of a point mass m located in a space referred to a coordinate system $x y z$ which rotates about the z axis with a constant angular velocity. Aside from the centrifugal and Coriolis forces, the mass is assumed to be under the influence of a force derivable from a particular potential function $\phi(x, y, z)$. It is further assumed that the motion of the mass m does not affect the location of the potential sources or the motion of the remainder of the system.

The centrifugal and Coriolis forces arise from the necessity of considering the acceleration of m with respect to some fixed frame of reference. Matrix notation is used in Section II to derive the differential equations of motion by a transformation of Newton's equation of motion. The resulting differential equations involve partial derivatives of the potential function $\phi(x, y, z)$. Only special types of potential functions produce differential equations which are linear.

A particular potential function giving a set of linear differential equations is discussed in Section III. This may be considered as a "generalized spring" problem in which the mass m is connected to n fixed anchor points (x_1, y_1, z_1) by means of n linear springs, each of a

particular stiffness k_1 . It is shown that the resulting motion of m is the same as that which would occur if the entire system of springs and anchor points were replaced by an equivalent single spring attached to an equivalent single anchor point. Explicit solutions are obtained by use of the Laplace transformation.

Following this linear problem, the remainder of the paper deals with potential functions associated with the inverse square law of gravitational attraction. Section IV concerns libration points, the Jacobian integral, and the expression of the equations of motion in abbreviated form.

Due to the non-linear nature of the differential equations associated with the gravitational potential function, the scope of the investigation is subsequently limited to the study of the motion of the mass m in the neighborhood of a libration point. In Section V the Laplace transformation is again used to establish a criterion for the stability of infinitesimal orbits about a libration point. Three cases are used in illustrating different sources of instability.

The stability of an infinitesimal orbit about a libration point requires that certain conditions be imposed on the potential function ϕ . Under a continuous variation of the geometry of a potential-producing system, the stability conditions may change. This is illustrated in

Section VI where three different systems are examined with regard to the stability criterion, the first of which is the well-known restricted problem of three bodies. The second is a restricted problem of four bodies and the third is a problem of a point mass centered on a rotating, infinite rod.

The paper is concluded with Section VII in which appears a more extended study of the latter problem, following a method previously published in connection with the restricted problem of three bodies. This involves the use of higher order approximations and gives additional information regarding the shape of the orbit and its angular frequency.

The results presented in this paper represent some particular aspects of the study of orbital motion of a mass in a potential field. Many problems of this general character appear in mathematical works of historical significance. Related material may be found in the literature cited in the bibliography of this paper.

II DERIVATION OF EQUATIONS OF MOTION

As previously mentioned, this paper concerns the motion of a point mass m , whose coordinates are referred to an orthogonal xyz -coordinate system which has a rotation with respect to a stationary, orthogonal $x_0y_0z_0$ -coordinate system. The z axis coincides with the z_0 axis and the xyz -system rotates about the z axis with a constant angular velocity ω . In addition to the forces associated with rotation, the mass m is also acted upon by a force derivable from a potential function $\phi(x,y,z)$.

If the stationary system $x_0y_0z_0$ is initially coincident with the rotating system xyz , and t represents the time, the coordinate transformation may be indicated by the matrix notation,

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (2.01)$$

The transformation may be more compactly written as

$$X_0 = BX \quad (2.02)$$

where B is the matrix indicated above.

The velocity transformation is obtained by differentiating (2.02) with respect to t ,

$$\frac{dX_0}{dt} = B \frac{dX}{dt} + \frac{dB}{dt} X \quad (2.03)$$

The acceleration transformation is obtained by differentiating (2.03) again with respect to t ,

$$\frac{d^2 X_0}{dt^2} = B \frac{d^2 X}{dt^2} + 2 \frac{dB}{dt} \frac{dX}{dt} + \frac{d^2 B}{dt^2} X. \quad (2.04)$$

By performing the operations indicated in (2.04) and using the dot to denote differentiation with respect to t , one obtains

$$\begin{pmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2\omega \dot{y} - \omega^2 x \\ \ddot{y} + 2\omega \dot{x} - \omega^2 y \\ \ddot{z} \end{pmatrix}. \quad (2.05)$$

The comparison of (2.05) with (2.01) shows that (2.05) may be written

$$\ddot{X}_0 = BP, \quad (2.06)$$

where P represents the column vector,

$$P = \begin{pmatrix} \ddot{x} - 2\omega \dot{y} - \omega^2 x \\ \ddot{y} + 2\omega \dot{x} - \omega^2 y \\ \ddot{z} \end{pmatrix}. \quad (2.07)$$

Since both the rotating system and the stationary system of coordinates are orthogonal, the inverse B^{-1} of the matrix B is the same¹ as its transpose B' . Multiplication of (2.02) by B' gives

$$B'X_0 = X, \quad (2.08)$$

and multiplication of (2.06) by B' gives

$$B'\ddot{X}_0 = P. \quad (2.09)$$

Reference to Equations (2.08) and (2.09) shows that the matrix B' required to transform the position vector X_0

¹Birkhoff, Garrett and MacLane, Saunders. A Survey of Modern Algebra. New York. Macmillan. p. 225. c1941.

into the vector X must also transform the acceleration vector \ddot{X}_0 into the vector P , where the components of P are given by (2.07).

With respect to the stationary coordinate system $x_0 y_0 z_0$, the vector form of Newton's equation of motion is,

$$F_0 = m \ddot{X}_0, \quad (2.10)$$

where the vector force F_0 acting on the mass m has a component in each of the directions x_0 , y_0 , and z_0 .

Equation (2.10) may be multiplied on the left by the matrix B' to give ,

$$B'F_0 = m B' \ddot{X}_0. \quad (2.11)$$

By use of (2.09), the above equation may be written,

$$F = m P, \quad (2.12)$$

where,

$$F = B'F_0. \quad (2.13)$$

The transformation of the vector F_0 gives the force vector F with components in the x , y , and z directions. If these components are f_1 , f_2 , and f_3 respectively, then (2.07) and (2.12) give the three equations,

$$f_1 = m (\ddot{x} - 2\omega\dot{y} - \omega^2x) \quad (2.14)$$

$$f_2 = m (\ddot{y} + 2\omega\dot{x} - \omega^2y) \quad (2.15)$$

$$f_3 = m \ddot{z} \quad (2.16)$$

These results are identical with those obtained by other means.²

²Brand, Louis. Vector and Tensor Analysis. New York. John Wiley and Sons. p.123. 1947.

This paper is concerned only with those cases in which the force vector F in (2.12) is the gradient of a scalar potential function $\phi(x, y, z)$. Such a function gives rise to three differential equations corresponding to (2.14), (2.15), and (2.16). The equations of motion thus obtained are,

$$m \left(\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - \omega^2 x \right) = \frac{\partial \phi}{\partial x} \quad (2.17)$$

$$m \left(\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} - \omega^2 y \right) = \frac{\partial \phi}{\partial y} \quad (2.18)$$

$$m \frac{d^2 z}{dt^2} = \frac{\partial \phi}{\partial z} . \quad (2.19)$$

Many types of potential functions, such as logarithmic and inverse-distance functions, produce differential equations which are non-linear and incapable of solution in terms of tabulated functions by known methods. Most of the results presented here will deal with non-linear equations arising from a particular type of potential function, namely, that due to gravitational attraction. As an example of a potential function giving rise to linear differential equations, a "generalized spring" problem will first be discussed. This is the problem of n force centers, or centers of attraction, with the force directly proportional to displacement.

III. PROBLEM OF n FORCE CENTERS WITH FORCE DIRECTLY PROPORTIONAL TO DISPLACEMENT

Consider n centers of attraction located at the points P_i with coordinates (x_i, y_i, z_i) , ($i = 1, 2, \dots, n$), in a rectangular coordinate system which rotates about the z axis with a constant angular velocity ω . Suppose a point mass m located at (x, y, z) is attracted toward each force center P_i by a force proportional to its displacement r_i from P_i . Let the constant of proportionality be k_i . The potential ϕ due to the location of m may then be written,

$$\phi = - \sum_{i=1}^n \frac{1}{2} k_i r_i^2, \quad (3.01)$$

where,

$$r_i^2 = (x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2. \quad (3.02)$$

The partial derivatives ϕ_x , ϕ_y , and ϕ_z of this potential function are,

$$\phi_x = \frac{\partial \phi}{\partial x} = \sum_{i=1}^n k_i (x_i - x) \quad (3.03)$$

$$\phi_y = \frac{\partial \phi}{\partial y} = \sum_{i=1}^n k_i (y_i - y) \quad (3.04)$$

$$\phi_z = \frac{\partial \phi}{\partial z} = \sum_{i=1}^n k_i (z_i - z) \quad (3.05)$$

The equations of motion (2.17), (2.18), and (2.19) then become,

$$m \left(\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - \omega^2 x \right) = \sum_{i=1}^n k_i (x_i - x) \quad (3.06)$$

$$m \left(\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} - \omega^2 y \right) = \sum_{i=1}^n k_i (y_i - y) \quad (3.07)$$

$$m \frac{d^2 z}{dt^2} = \sum_{i=1}^n k_i (z_i - z). \quad (3.08)$$

By using a dot to denote differentiation with respect to t the above equations may be rewritten in the form,

$$\ddot{x} + (b^2 - \omega^2)x - 2\omega \dot{y} = a_1 \quad (3.09)$$

$$\ddot{y} + (b^2 - \omega^2)y + 2\omega \dot{x} = a_2 \quad (3.10)$$

$$\ddot{z} + b^2 z = a_3, \quad (3.11)$$

where,

$$a_1 = \frac{1}{m} \sum_{i=1}^n k_i x_i \quad (3.12)$$

$$a_2 = \frac{1}{m} \sum_{i=1}^n k_i y_i \quad (3.13)$$

$$a_3 = \frac{1}{m} \sum_{i=1}^n k_i z_i \quad (3.14)$$

$$b^2 = \frac{1}{m} \sum_{i=1}^n k_i . \quad (3.15)$$

Equation (3.11) may be solved independently of the other two equations of motion to give,

$$z = \frac{a_3}{b^2} + \left(z_0 - \frac{a_2}{b^2} \right) \cos bt + \frac{\dot{z}_0}{b} \sin bt, \quad (3.16)$$

where the subscript $_0$ following a variable denotes the initial value of that variable.

Equations (3.09) and (3.10) may be solved by use of the Laplace transformation. The transforms of these equations are,

$$(s^2 + c^2)x(s) - 2\omega sy(s) = \frac{a_1}{s} + \dot{x}_0 - 2\omega y_0 + x_0 s \quad (3.17)$$

$$2\omega sx(s) + (s^2 + c^2)y(s) = \frac{a_2}{s} + \dot{y}_0 + 2\omega x_0 + y_0 s \quad (3.18)$$

where $x(s)$ and $y(s)$ are the transforms of $x(t)$ and $y(t)$ and

$$c^2 = b^2 - \omega^2 . \quad (3.19)$$

Equations (3.17) and (3.18) may be solved for the transforms $x(s)$ and $y(s)$. The determinant of the coefficients is,

$$D(s) = s^4 + 2(c^2 + 2\omega^2)s^2 + c^4 . \quad (3.20)$$

In view of (3.19), this determinant may be written,

$$D(s) = s^4 + 2(b^2 + \omega^2)s^2 + (b^2 - \omega^2)^2 . \quad (3.21)$$

Equation (3.21) will be factored for later use.

Thus,

$$D(s) = [s^2 + (b + \omega)^2] [s^2 + (b - \omega)^2] \quad (3.22)$$

The use of Cramer's rule in solving (3.17) and (3.18) for $x(s)$ and $y(s)$ is shown in detail in Appendix A. The transforms thus found are,

$$x(s) = \frac{A_3 s^3 + A_2 s^2 + A_1 s + A_0 + A_{-1} s^{-1}}{[s^2 + (b + \omega)^2] [s^2 + (b - \omega)^2]} \quad (3.23)$$

$$y(s) = \frac{B_3 s^3 + B_2 s^2 + B_1 s + B_0 + B_{-1} s^{-1}}{[s^2 + (b + \omega)^2] [s^2 + (b - \omega)^2]} \quad (3.24)$$

The constants appearing in these equations are determined by the initial conditions and the parameters of the system.

$$A_3 = x_0 \quad (3.25)$$

$$A_2 = \dot{x}_0 \quad (3.26)$$

$$A_1 = a_1 + 2 \dot{y}_0 + (b^2 + 3\omega^2)x_0 \quad (3.27)$$

$$A_0 = (b^2 - \omega^2)(\dot{x}_0 - 2\omega y_0) + 2\omega a_2 \quad (3.28)$$

$$A_{-1} = a_1(b^2 - \omega^2) \quad (3.29)$$

$$B_3 = y_0 \quad (3.30)$$

$$B_2 = \dot{y}_0 \quad (3.31)$$

$$B_1 = a_2 - 2\omega \dot{x}_0 + (b^2 + 3\omega^2)y_0 \quad (3.32)$$

$$B_0 = (b^2 - \omega^2)(\dot{y}_0 + 2\omega x_0) - 2\omega a_1 \quad (3.33)$$

$$B_{-1} = a_2(b^2 - \omega^2) \quad (3.34)$$

In the expressions for $x(s)$ and $y(s)$, given by (3.23) and (3.24), it is seen that each term contains a factor

$f(s)$ where,

$$f(s) = \frac{1}{[s^2 + (b+\omega)^2][s^2 + (b-\omega)^2]} \quad (3.35)$$

It may readily be shown that if $b^2 \neq \omega^2$ then $f(s)$ is the transform of the function $F_1(t)$ where,

$$F_1(t) = \frac{(b+\omega)\sin(b-\omega)t - (b-\omega)\sin(b+\omega)t}{4b\omega(b^2 - \omega^2)} \quad (3.36)$$

In the singular case where $b^2 = \omega^2$ it may be shown that $f(s)$ is the transform of the function $F_2(t)$ where,

$$F_2(t) = \frac{2\omega t - \sin 2\omega t}{8\omega^3} \quad (3.37)$$

In obtaining the inversions $x(t)$ and $y(t)$ of $x(s)$ and $y(s)$ it is convenient to use the following properties³ of the Laplace transform. If $g(s)$ is the transform of $G(t)$ then,

$$s^3 g(s) = L\{G'''(t)\} + s^2 G(0) + sG'(0) + G''(0) \quad (3.38)$$

$$s^2 g(s) = L\{G''(t)\} + sG(0) + G'(0) \quad (3.39)$$

$$sg(s) = L\{G'(t)\} + G(0) \quad (3.40)$$

$$\frac{1}{s} g(s) = L\left\{\int_0^t G(\tau) d\tau\right\}, \quad (3.41)$$

where the prime denotes a derivative with respect to t .

The required initial values of $G(t)$, $G'(t)$, and $G''(t)$ are identically zero if $G(t)$ is either of the functions $F_1(t)$

³Churchill, Ruel V. Modern Operational Mathematics in Engineering. New York. McGraw-Hill. pp.8,39. 1944.

or $F_2(t)$ given by (3.36) and (3.37).

By using the above properties together with (3.35), (3.36), and (3.37), the inversions of (3.23) and (3.24) may be obtained. Details of the operations are shown in appendix B. The results for $b^2 \neq \omega^2$ are,

$$x(t) = C_1 + C_2 \sin(b+\omega)t + C_3 \cos(b+\omega)t + C_4 \sin(b-\omega)t + C_5 \cos(b-\omega)t \quad (3.42)$$

$$y(t) = D_1 + D_2 \sin(b+\omega)t + D_3 \cos(b+\omega)t + D_4 \sin(b-\omega)t + D_5 \cos(b-\omega)t, \quad (3.43)$$

where,

$$C_1 = \frac{a_1}{b^2 - \omega^2} \quad (3.44)$$

$$C_2 = \frac{b-\omega}{2b} y_0 + \frac{\dot{x}_0}{2b} - \frac{a_2}{2b(b+\omega)} \quad (3.45)$$

$$C_3 = \frac{b-\omega}{2b} x_0 - \frac{\dot{y}_0}{2b} - \frac{a_1}{2b(b+\omega)} \quad (3.46)$$

$$C_4 = -\frac{b+\omega}{2b} y_0 + \frac{\dot{x}_0}{2b} + \frac{a_2}{2b(b-\omega)} \quad (3.47)$$

$$C_5 = \frac{b+\omega}{2b} x_0 + \frac{\dot{y}_0}{2b} - \frac{a_1}{2b(b-\omega)} \quad (3.48)$$

$$D_1 = \frac{a_2}{b^2 - \omega^2} \quad (3.49)$$

$$D_2 = -\frac{b-\omega}{2b} x_0 + \frac{\dot{y}_0}{2b} + \frac{a_1}{2b(b+\omega)} \quad (3.50)$$

$$D_3 = \frac{b-\omega}{2b} y_0 + \frac{\dot{x}_0}{2b} - \frac{a_2}{2b(b+\omega)} \quad (3.51)$$

$$D_4 = \frac{b+\omega}{2b} x_0 + \frac{\dot{y}_0}{2b} - \frac{a_1}{2b(b-\omega)} \quad (3.52)$$

$$D_5 = \frac{b+\omega}{2b} y_0 - \frac{\dot{x}_0}{2b} - \frac{a_2}{2b(b-\omega)}. \quad (3.53)$$

The results for the singular case where $b^2 = \omega^2$ are,

$$x(t) = L_1 + L_2 t + L_3 \sin 2\omega t + L_4 \cos 2\omega t \quad (3.54)$$

$$y(t) = M_1 + M_2 t + M_3 \sin 2\omega t + M_4 \cos 2\omega t, \quad (3.55)$$

where,

$$L_1 = \frac{a_1 + 2\omega \dot{y}_0 + 4\omega^2 x_0}{4\omega^2} \quad (3.56)$$

$$L_2 = \frac{a_2}{2\omega} \quad (3.57)$$

$$L_3 = \frac{2\omega \dot{x}_0 - a_2}{4\omega^2} \quad (3.58)$$

$$L_4 = -\frac{2\omega \dot{y}_0 + a_1}{4\omega^2} \quad (3.59)$$

$$M_1 = \frac{a_2 - 2\omega \dot{x}_0 + 4\omega^2 y_0}{4} \quad (3.60)$$

$$M_2 = -\frac{a_1}{2\omega} \quad (3.61)$$

$$M_3 = \frac{2\omega y_0 + a_1}{4\omega^2} \quad (3.62)$$

$$M_4 = \frac{2\omega \dot{x}_0 - a_2}{4\omega^2} \quad (3.63)$$

Since each of the equations (3.54) and (3.55) contain a term in which t is a factor, it follows that these solutions do not remain bounded for all values of t unless the constants

a_1 and a_2 are zero.

An interesting result occurs when $b^2 = \omega^2$ and $a_1 = a_2 = a_3 = 0$, together with the condition that $z_0 = \dot{z}_0 = 0$. Then (3.16) shows that z remains zero for all values of t , and (3.54) and (3.55) reduce to the forms,

$$x - \left(x_0 + \frac{\dot{y}_0}{2\omega} \right) = \frac{\dot{x}_0}{2\omega} \sin 2\omega t - \frac{\dot{y}_0}{2\omega} \cos 2\omega t \quad (3.64)$$

$$y - \left(y_0 - \frac{\dot{x}_0}{2\omega} \right) = \frac{\dot{y}_0}{2\omega} \sin 2\omega t + \frac{\dot{x}_0}{2\omega} \cos 2\omega t \quad (3.65)$$

In this instance the mass m moves in a clockwise direction with constant velocity v_0 around a circle of radius $\frac{v_0}{2\omega}$ centered at the point $\left(x_0 + \frac{\dot{y}_0}{2\omega}, y_0 - \frac{\dot{x}_0}{2\omega} \right)$ in the $x y$ plane.

The foregoing discussion may be summarized by the use of a mechanical analogy. The problem is that of determining the motion of a mass m connected, by means of n linear springs each of stiffness k_i , to n fixed points (x_i, y_i, z_i) in a uniformly rotating coordinate system $x y z$. Reference to (3.09), (3.10), and (3.11) shows that this problem can be reduced to that of determining the motion of the mass m connected, by means of a single equivalent spring of stiffness K , to a single equivalent point (s_1, s_2, s_3) . The quantities s_1, s_2, s_3 , and K are given by,

$$s_1 = \sum_{i=1}^n k_i x_i \quad (3.66)$$

$$s_2 = \sum_{i=1}^n k_i y_i \quad (3.67)$$

$$s_3 = \sum_{i=1}^n k_i z_i \quad (3.68)$$

$$K = \sum_{i=1}^n k_i \quad (3.69)$$

IV. LIBRATION POINTS

Throughout the remainder of this paper the potential function ϕ will be the gravitational potential associated with the inverse square law of attraction. It will also be assumed that the mass distribution producing the potential function $\phi(x, y, z)$ is confined to the $x y$ -plane. The discussion presented here will deal only with motion in the $x y$ plane. All masses will be assumed to be point masses unless otherwise stated. The mass m whose motion is to be studied will be assumed infinitesimal in magnitude when compared to any other mass fixed in the uniformly rotating coordinate system $x y z$. Thus the potential ϕ influences the motion of m but is not disturbed by it.

Under the above assumptions the equations of motion of m derived in Section II become,

$$m \left(\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - \omega^2 x \right) = \frac{\partial \phi}{\partial x} \quad (4.01)$$

$$m \left(\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} - \omega^2 y \right) = \frac{\partial \phi}{\partial y} . \quad (4.02)$$

By using the substitution,

$$\tau = \omega t , \quad (4.03)$$

the above equations of motion may be written,

$$\frac{d^2 x}{d\tau^2} - 2 \frac{dy}{d\tau} - x = \frac{\partial}{\partial x} \left(\frac{\phi}{m\omega^2} \right) \quad (4.04)$$

$$\frac{d^2 y}{d\tau^2} + 2 \frac{dx}{d\tau} - y = \frac{\partial}{\partial y} \left(\frac{\phi}{m\omega^2} \right) \quad (4.05)$$

Equations (4.04) and (4.05) may be written,

$$\frac{d^2 x}{d\tau^2} - 2 \frac{dy}{d\tau} = \frac{\partial}{\partial x} \left(\frac{r^2}{2} + \frac{\phi}{m\omega^2} \right) \quad (4.06)$$

$$\frac{d^2 y}{d\tau^2} + 2 \frac{dx}{d\tau} = \frac{\partial}{\partial y} \left(\frac{r^2}{2} + \frac{\phi}{m\omega^2} \right), \quad (4.07)$$

where,

$$r^2 = x^2 + y^2. \quad (4.08)$$

The above equations of motion may be written more compactly by using subscripts to denote partial derivatives and a dot to indicate a derivative with respect to τ . Thus,

$$\ddot{x} - 2\dot{y} = \Omega_x \quad (4.09)$$

$$\ddot{y} + 2\dot{x} = \Omega_y, \quad (4.10)$$

where,

$$\Omega = \frac{r^2}{2} + \frac{\phi}{m\omega^2}. \quad (4.11)$$

Equations (4.09) and (4.10) are the equations of motion which will be used in the remainder of this paper.

Points in the $x y$ plane where Ω_x and Ω_y both vanish are known as libration points. The mass m may remain at rest in the $x y$ plane only at a libration point. It may be shown that if the mass m passes through a libration point its acceleration (with respect to the $x y$ coordinate system) must be perpendicular to its path at the instant of transit.

Libration points are located⁴ by solving the equations,

$$\Omega_x = 0 \quad (4.12)$$

$$\Omega_y = 0 \quad (4.13)$$

It may readily be shown that equations (4.09) and (4.10) admit the Jacobian integral,

$$v^2 = 2\Omega - C, \quad (4.14)$$

where C is a constant of integration and v is the velocity defined by,

$$v^2 = \dot{x}^2 + \dot{y}^2. \quad (4.15)$$

It was previously mentioned that in general the gravitational potential produces differential equations which are of a non-linear nature and often incapable of solution in terms of the elementary functions by known methods. However, by restricting the scope of the problem some useful results may be obtained. In the following the problem will be restricted to the study of motion in the neighborhood of a libration point.

Suppose, corresponding to a certain function Ω defined by (4.11), a libration point L is located at (a, b) in the $x y$ plane by solving equations (4.12) and (4.13). The function Ω is assumed to possess a valid Taylor's series

⁴Hinrichsen, J. J. L. The Libration Points in an n -body Problem. American Mathematical Monthly. 50, no. 4: 231-237. April 1943.

expansion about this point. By using the substitutions,

$$\xi = x - a \quad (4.16)$$

$$\eta = y - b, \quad (4.17)$$

Equations (4.09) and 4.10) may be written,

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{\xi}(\xi, \eta) \quad (4.18)$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{\eta}(\xi, \eta). \quad (4.19)$$

Since the partial derivatives Ω_{ξ} and Ω_{η} vanish at the libration point, which is at the origin of the $\xi\eta$ plane, their expansions are,

$$\Omega_{\xi}(\xi, \eta) = \xi \Omega_{\xi\xi}(0,0) + \eta \Omega_{\xi\eta}(0,0) + \frac{\xi^2}{2} \Omega_{\xi\xi\xi}(0,0) + \dots \quad (4.20)$$

$$\Omega_{\eta}(\xi, \eta) = \xi \Omega_{\eta\xi}(0,0) + \eta \Omega_{\eta\eta}(0,0) + \frac{\xi^2}{2} \Omega_{\eta\xi\xi}(0,0) + \dots \quad (4.21)$$

There now arises the question whether the infinitesimal mass m , if initially placed near the libration point L , would remain in the neighborhood of L in spite of a small disturbance from a source isolated from the system. This is a question of the stability of infinitesimal orbits in the neighborhood of a libration point. The criterion for this stability will next be established and illustrated with examples.

V. STABILITY OF INFINITESIMAL ORBITS ABOUT LIBRATION POINTS

Let ξ and η represent the displacements in the x and y directions of the infinitesimal mass m from the libration point L . For ξ and η small, the expansions of Ω_ξ and Ω_η given by (4.20) and (4.21) may be approximated by retaining only the linear terms. Then, from (4.18) and (4.19), the equations of motion may be approximated by,

$$\ddot{\xi} - 2\dot{\eta} = \xi \Omega_{\xi\xi} + \eta \Omega_{\xi\eta} \quad (5.01)$$

$$\ddot{\eta} + 2\dot{\xi} = \xi \Omega_{\eta\xi} + \eta \Omega_{\eta\eta}, \quad (5.02)$$

where the partial derivatives of Ω are evaluated at the libration point and are thus constants. All derivatives of will be assumed continuous in the neighborhood of the libration point L .

In order to introduce the initial conditions immediately, the Laplace transformation with respect to τ will be used. The transforms of (5.01) and (5.02) are,

$$(s^2 - \Omega_{\xi\xi}) \xi(s) - (2s + \Omega_{\xi\eta}) \eta(s) = s \xi_0 + \dot{\xi}_0 - 2 \eta_0 \quad (5.03)$$

$$(2s - \Omega_{\eta\xi}) \xi(s) + (s^2 - \Omega_{\eta\eta}) \eta(s) = s \eta_0 + \dot{\eta}_0 + 2 \xi_0, \quad (5.04)$$

where $\xi(s)$ and $\eta(s)$ are the transforms of $\xi(\tau)$ and $\eta(\tau)$ and ξ_0 , $\dot{\xi}_0$, η_0 , and $\dot{\eta}_0$ are the initial values of ξ , $\dot{\xi}$, η , and $\dot{\eta}$.

The determinant of the coefficients in (5.03) and (5.04)

is,

$$D(s) = s^4 - (\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4)s^2 + (\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2). \quad (5.05)$$

This may be written in the factored form,

$$D(s) = (s^2 - s_1^2)(s^2 - s_2^2), \quad (5.06)$$

where,

$$2s_1^2 = (\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4) + \sqrt{(\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4)^2 - 4(\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2)} \quad (5.07)$$

$$2s_2^2 = (\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4) - \sqrt{(\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4)^2 - 4(\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2)}. \quad (5.08)$$

The solution of (5.03) and (5.04) for the transforms

$\xi(s)$ and $\eta(s)$ gives,

$$\xi(s) = \frac{p_3 s^3 + p_2 s^2 + p_1 s + p_0}{(s^2 - s_1^2)(s^2 - s_2^2)} \quad (5.09)$$

$$\eta(s) = \frac{q_3 s^3 + q_2 s^2 + q_1 s + q_0}{(s^2 - s_1^2)(s^2 - s_2^2)}, \quad (5.10)$$

where,

$$p_3 = \dot{\xi}_0 \quad (5.11)$$

$$p_2 = \ddot{\xi}_0 \quad (5.12)$$

$$p_1 = (4 - \Omega_{\eta\eta})\dot{\xi}_0 + \Omega_{\xi\eta}\dot{\eta}_0 + 2\dot{\xi}_0 \quad (5.13)$$

$$p_0 = \Omega_{\xi\eta}(\dot{\eta}_0 + 2\dot{\xi}_0) - \Omega_{\eta\eta}(\ddot{\xi}_0 - 2\dot{\eta}_0) \quad (5.14)$$

$$q_3 = \dot{\eta}_0 \quad (5.15)$$

$$q_2 = \ddot{\eta}_0 \quad (5.16)$$

$$q_1 = (4 - \Omega_{\xi\xi})\dot{\eta}_0 + \Omega_{\xi\eta}\dot{\xi}_0 - 2\dot{\xi}_0 \quad (5.17)$$

$$q_0 = \Omega_{\xi\eta}(\ddot{\xi}_0 - 2\dot{\eta}_0) - \Omega_{\xi\xi}(\dot{\eta}_0 + 2\dot{\xi}_0) \quad (5.18)$$

The equations whose transforms are given by (5.09) and (5.10) define approximately the motion of m while it is in the immediate neighborhood of the libration point L . For a given set of initial conditions the infinitesimal mass m moves in an orbit which may or may not be enclosed by a small circle of finite radius centered at L . This infinitesimal orbit will be called stable if it always remains within the circle. The orbit will be called unstable if it does not remain within such a circle.

Since this discussion is valid only for an orbit in the neighborhood of L , the initial conditions must be commensurate with an infinitesimal orbit. In addition to the conditions established below it will be assumed that the initial conditions by themselves will not produce an orbit which exceeds infinitesimal dimensions. The initial conditions are otherwise considered arbitrary.

The functions obtained by inverting (5.09) and (5.10) depend upon the character of their common denominator $D(s)$. It is evident that for the resulting orbit to be stable the inversions $\xi(\tau)$ and $\eta(\tau)$ of $\xi(s)$ and $\eta(s)$ must be functions which remain bounded. Thus, the stability conditions can be determined by examining the zeros of $D(s)$.

By considering the possible inversions of (5.09) and (5.10) it is seen that stable orbits exist if and only if s_1^2 and s_2^2 are negative, real, and unequal. These

requirements are satisfied by the necessary and sufficient conditions,

$$\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4 < 0 \quad (5.19)$$

$$\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2 > 0 \quad (5.20)$$

$$4(\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2) < (\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4)^2. \quad (5.21)$$

Some special cases leading to conditions of instability will now be discussed. These are limiting cases which justify the omission of equal signs under the inequality signs in (5.19), (5.20), and (5.21).

First, suppose that,

$$\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4 = 0. \quad (5.22)$$

The expression for $D(s)$ given by (5.05) then becomes,

$$D(s) = s^4 + (\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2). \quad (5.23)$$

As a result of this denominator in $\xi(s)$ and $\eta(s)$ the inversions of (5.09) and (5.10) contain hyperbolic sines and cosines of τ . Hence, the infinitesimal orbits are unstable in this case.

Next, suppose that,

$$\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2 = 0. \quad (5.24)$$

It then follows from (5.08) that s_2 vanishes and the expression for $D(s)$ becomes,

$$D(s) = s^2(s^2 - s_1^2). \quad (5.25)$$

As a result of this denominator in $\xi(s)$ and $\eta(s)$ the inversions of (5.09) and (5.10) contain terms having τ as a factor. The resulting orbits are therefore unstable.

Finally, if

$$(\Omega_{\xi\xi} + \Omega_{\eta\eta} - 4)^2 = 4(\Omega_{\xi\xi}\Omega_{\eta\eta} - \Omega_{\xi\eta}^2) , \quad (5.26)$$

then s_1^2 and s_2^2 are equal and $D(s)$ becomes,

$$D(s) = (s^2 + s_1^2)^2 . \quad (5.27)$$

as a result of this denominator in $\xi(s)$ and $\eta(s)$ the inversions of (5.09) and (5.10) contain terms having τ as a factor. Thus, the resulting orbits are unstable.

Other cases not satisfying conditions (5.19), (5.20), and (5.21) may likewise be shown to lead to unstable infinitesimal orbits.

In view of the substitutions given by (4.16) and (4.17), the question of stability of infinitesimal orbits in the neighborhood of a libration point may be answered by the following theorem.

Theorem. Suppose for some function of $\Omega(x, y)$ a libration point L is located at (a, b) in the xy -plane. Necessary and sufficient conditions for stable infinitesimal orbits about L are:

$$\Omega_{xx} + \Omega_{yy} - 4 < 0 \quad (5.28)$$

$$\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 > 0 \quad (5.29)$$

$$4(\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2) < (\Omega_{xx} + \Omega_{yy} - 4)^2 , \quad (5.30)$$

where the partial derivatives of Ω are evaluated at (a, b) .

The conditions (5.28), (5.29, and (5.30) are somewhat analogous to a condition of equilibrium. A mass m placed

at a libration point L may be considered as being in a state of equilibrium, since it will remain at L in the absence of disturbances from sources external to the system. This state of equilibrium is unstable if conditions (5.28), (5.29), and (5.30) are not satisfied, since the slightest disturbance would cause the mass m to follow a path which may not lie within the neighborhood of L .

Applications of the stability criterion will now be illustrated with several examples, the first of which is the well-known restricted problem of three bodies whose physical analogy is the system⁵ consisting of the Sun, Jupiter, and the Trojan Asteroids.

⁵Brown, Ernest W. and Shook, Clarence A. Planetary Theory. London. Cambridge University Press. p.250. 1933.

VI. APPLICATIONS OF THE STABILITY CRITERION

A. The Restricted Problem of Three Bodies

The potential ϕ in this problem is produced by two finite masses m_1 and m_2 located at $(-c_1, 0)$ and $(c_2, 0)$ in the $x y$ plane as shown in Figure 1.

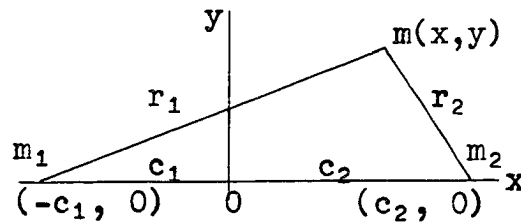


Fig. 1. Restricted Problem of Three Bodies.

If c is the distance between m_1 and m_2 , and γ is the gravitational constant, the result of equating forces acting on m_1 and m_2 is,

$$m_1 \omega^2 c_1 = m_2 \omega^2 c_2 = \gamma \frac{m_1 m_2}{c^2} \quad (6.01)$$

where ω is the constant angular velocity of rotation of the system. If r_1 and r_2 are the distances from the infinitesimal mass m to m_1 and m_2 , respectively, the potential ϕ is given by,

$$\phi = \gamma m \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right), \quad (6.02)$$

where,

$$r_1^2 = (x+c_1)^2 + y^2 \quad (6.03)$$

$$r_2^2 = (x-c_2)^2 + y^2 \quad (6.04)$$

In view of (6.01), the potential ϕ given by (6.02) may be written,

$$\phi = m\omega^2 c^2 \left(\frac{c_2}{r_1} + \frac{c_1}{r_2} \right) \quad (6.05)$$

The equations of motion (4.09) and (4.10) remain valid for an arbitrary choice of length unit in the expression for ϕ . For this problem a length unit will be chosen such that the distance c between m_1 and m_2 is unity. The distances c_1 and c_2 may then be written,

$$c_1 = \mu \quad (6.06)$$

$$c_2 = 1 - \mu, \quad (6.07)$$

where μ is a positive number less than unity. After making these substitutions in (6.05) one obtains

$$\phi = m\omega^2 \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right), \quad (6.08)$$

where,

$$r_1^2 = (x+\mu)^2 + y^2 \quad (6.09)$$

$$r_2^2 = (x+\mu-1)^2 + y^2 \quad (6.10)$$

The function Ω defined by (4.11) is then,

$$\Omega = \frac{r^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \quad (6.11)$$

Many of the earlier investigations of the restricted problem of three bodies were concerned with the libration

points of which there are five. Three lie on the x-axis, collinear with the masses m_1 and m_2 , while two lie at the points $\left(\frac{1}{2} - \mu, \pm \frac{\sqrt{3}}{2}\right)$. The latter points, L_1 and L_2 , lie in the vertices of equilateral triangles, each of which has the line joining m_1 to m_2 as a base. These triangular configurations are observed in practice where m_1 is the sun, m_2 is the planet Jupiter, and the neighborhoods of L_1 and L_2 are occupied by the Trojan asteroids.

The conditions for the existence of stable infinitesimal orbits in the neighborhood of the libration point L_1 at $\left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right)$ may be obtained by applying the stability criterion to (6.11). The required partial derivatives of Ω are,

$$\Omega_{xx}(x,y) = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 3(1-\mu)\frac{(x+\mu)^2}{r_1^5} + 3\mu\frac{(x+\mu-1)^2}{r_2^5} \quad (6.12)$$

$$\Omega_{yy}(x,y) = 1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 3(1-\mu)\frac{y^2}{r_1^5} + 3\mu\frac{y^2}{r_2^5} \quad (6.13)$$

$$\Omega_{xy}(x,y) = 3(1-\mu)\frac{(x+\mu)y}{r_1^5} + 3\mu\frac{(x+\mu-1)y}{r_2^5}. \quad (6.14)$$

At the libration point L_1 these partial derivatives become,

$$\Omega_{xx}\left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right) = \frac{3}{4} \quad (6.15)$$

$$\Omega_{yy}\left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right) = \frac{9}{4} \quad (6.16)$$

$$\Omega_{xy}\left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}(1-2\mu). \quad (6.17)$$

When the above values are substituted into the inequalities (5.28), (5.29, and (5.30) it is seen that the conditions for stable infinitesimal orbits are satisfied if,

$$0 < 27\mu(1-\mu) < 1. \quad (6.18)$$

By substituting (6.06) and (6.07) into (6.01), it is found that,

$$\mu = \frac{m_2}{m_1 + m_2} \quad (6.19)$$

$$1 - \mu = \frac{m_1}{m_1 + m_2} . \quad (6.20)$$

The substitution of (6.19) and (6.20) into (6.18) gives,

$$0 < \frac{\sqrt{m_1 m_2}}{m_1 + m_2} < \frac{\sqrt{3}}{9} . \quad (6.21)$$

It may be readily shown that a libration point exists at the point $\left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}\right)$ in the $x y$ plane for all positive values of m_1 and m_2 . However, it is seen from (6.21) that the existence of stable infinitesimal orbits in the neighborhood of this point requires m_1 and m_2 to be such that the ratio of their geometric mean to their arithmetic mean is greater than zero but less than $\frac{2\sqrt{3}}{9}$.

B. A Restricted Problem of Four Bodies

In this problem the potential ϕ is produced by a system consisting of two equal masses located on the x axis at equal distances from a third mass m_2 fixed at the origin.

Let m_1 be the magnitude of each of the two equal masses and a_1 the distance of each from the origin as in Figure 2.

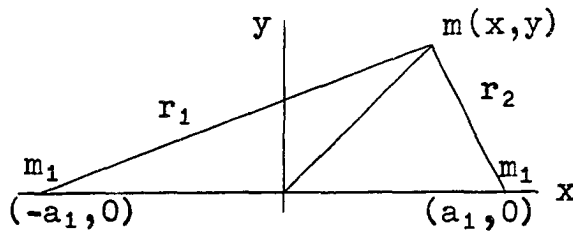


Fig. 2. A Restricted Problem of Four Bodies.

Let r_1 and r_2 denote the distances from the infinitesimal mass m to the two equal masses, and r the distance from m to the origin.

$$r_1^2 = (x+a_1)^2 + y^2 \quad (6.22)$$

$$r_2^2 = (x-a_1)^2 + y^2 \quad (6.23)$$

$$r^2 = x^2 + y^2 \quad (6.24)$$

The relation between the finite masses and the length a_1 is found by equating centrifugal and attractive forces. Thus,

$$m_1 \omega^2 a_1 = \gamma \frac{m_1 m_2}{a_1^2} + \gamma \frac{m_1^2}{4a_1^2}, \quad (6.25)$$

where γ is the gravitational constant and ω is the angular velocity of the system. By introducing the ratio,

$$\rho = \frac{m_1}{m_2}, \quad (6.26)$$

Equation (6.25) may be written,

$$\gamma \frac{m_2}{\omega^2} = \frac{4a_1^3}{4+\rho}. \quad (6.27)$$

The potential ϕ produced by the three finite masses is,

$$\phi = \gamma m \left(\frac{m_1}{r_1} + \frac{m_1}{r_2} + \frac{m_2}{r} \right). \quad (6.28)$$

By substituting (6.27) into (6.28), the potential function ϕ may be written,

$$\phi = \frac{4m\omega^2 a_1^3}{4+\rho} \left(\frac{\rho}{r_1} + \frac{\rho}{r_2} + \frac{1}{r} \right). \quad (6.29)$$

A length unit may now be chosen such that,

$$a_1^3 = 1 + \frac{\rho}{4}, \quad (6.30)$$

in which case the potential function becomes,

$$\phi = m \omega^2 \left(\frac{\rho}{r_1} + \frac{\rho}{r_2} + \frac{1}{r} \right). \quad (6.31)$$

The term a_1 in the expressions for r_1 and r_2 is obtained from (6.30). The function Ω defined by (4.11) is then,

$$\Omega = \frac{r^2}{2} + \frac{1}{r} + \frac{\rho}{r_1} + \frac{\rho}{r_2}. \quad (6.32)$$

The first partial derivatives of are,

$$\Omega_x(x,y) = x - \frac{x}{r^3} - \frac{\rho(x+a_1)}{r_1^3} - \frac{\rho(x-a_1)}{r_2^3} \quad (6.33)$$

$$\Omega_y(x,y) = y - \frac{y}{r^3} - \frac{\rho y}{r_1^3} - \frac{\rho y}{r_2^3}. \quad (6.34)$$

From these equations it is evident that libration points exist at $(0, \pm b_1)$ in the $x y$ plane, where the positive value of b_1 is obtained from the equation,

$$1 - \frac{2}{r_1^3} - \frac{1}{b_1^3} = 0. \quad (6.35)$$

By using the substitution,

$$b_1 = a_1 \tan \theta, \quad (6.36)$$

Equation (6.35) may be written,

$$\frac{1 - \cot^3 \theta}{8 \cos^3 \theta - 1} = \frac{\rho}{4}. \quad (6.37)$$

Inspection of Equation (6.37) shows that for ρ positive and finite, θ must be somewhere in the range,

$$\frac{\pi}{4} < \theta < \frac{\pi}{3}. \quad (6.38)$$

Equation (6.37) may be used to graphically locate the libration point L_1 at $(0, b_1)$ in the $x y$ -plane.

The conditions for the existence of stable infinitesimal orbits in the neighborhood of L_1 may be obtained by applying the stability criterion to (6.32). The required partial derivatives of Ω are,

$$\Omega_{xx}(x,y) = 1 - \frac{\rho}{r_1^3} - \frac{\rho}{r_2^3} - \frac{1}{r^3} + 3\rho \frac{(x+a_1)^2}{r_1^5} + 3\rho \frac{(x-a_1)^2}{r_2^5} \quad (6.39)$$

$$\begin{aligned} \Omega_{yy}(x,y) = & 1 - \frac{\rho}{r_1^3} - \frac{\rho}{r_2^3} - \frac{1}{r^3} + 3\rho \frac{y^2}{r_1^5} + 3\rho \frac{y^2}{r_2^5} \\ & + 3 \frac{x^2}{r^5} \end{aligned} \quad (6.40)$$

$$\Omega_{xy}(x,y) = 3\rho \frac{(x+a_1)y}{r_1^5} + 3\rho \frac{(x-a_1)y}{r_2^5} + 3 \frac{XY}{r^5} \quad (6.41)$$

These partial derivatives are evaluated at the libration point $(0, b_1)$ with the aid of (6.35). The results are,

$$\Omega_{xx}(0, b_1) = 6\rho \frac{a_1^2}{r_1^2} \quad (6.42)$$

$$\Omega_{yy}(0, b_1) = 6\rho \frac{b_1^2}{r_1^5} + \frac{3}{b_1^3} \quad (6.43)$$

$$\Omega_{xy}(0, b_1) = 0, \quad (6.44)$$

where,

$$r_1^2 = a_1^2 + b_1^2. \quad (6.45)$$

The quantities required in the stability criterion are found from the above equations with the aid of (6.35),

$$\Omega_{xx} + \Omega_{yy} - 4 = -1 \quad (6.46)$$

$$\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = \frac{18\rho a_1^2}{r_1^5} \left(1 - \frac{2\rho a_1^2}{r_1^5} \right). \quad (6.47)$$

From the inequalities (5.28), (5.29), and (5.30), it is seen that the conditions for stable infinitesimal orbits are satisfied if,

$$0 < v(1-2v) < \frac{1}{72}, \quad (6.48)$$

where,

$$v = \rho \frac{a_1^2}{r_1^5}. \quad (6.49)$$

Since v is a positive quantity, it may be shown that the inequality (6.48) is satisfied only if v is contained

within one of two ranges, namely,

$$0 < v < \frac{3-2\sqrt{2}}{12} \quad (6.50)$$

$$\frac{3+2\sqrt{2}}{12} < v < \frac{1}{2} . \quad (6.51)$$

One of these ranges may be eliminated by the method which follows.

By using (6.36) and (6.37), the expression for v given by (6.49) may be written,

$$v = g(\theta), \quad (6.52)$$

where,

$$g(\theta) = \frac{4(\sin^3\theta - \cos^3\theta)\cos^2\theta}{8\sin^3\theta - 1} . \quad (6.53)$$

The function $g(\theta)$ may be tabulated as in Table 1, and plotted as in Figure 3. Only the range of θ indicated by (6.38) is required, since this range includes all positive values of ρ . The graph of $g(\theta)$ shows that the maximum value of the function is less than 0.15. Thus, it is evident that the range of v indicated by (6.51) is eliminated.

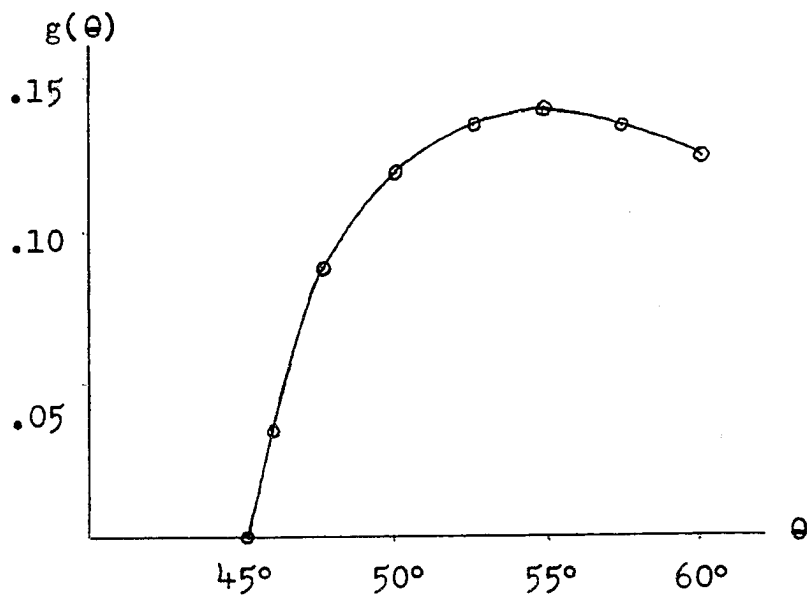
By using (6.52), the remaining condition (6.50) for stable infinitesimal orbits may be written,

$$0 < g(\theta) < \frac{3-2\sqrt{2}}{12} , \quad (6.54)$$

where $g(\theta)$ is defined by (6.53) and θ is defined by (6.36). Reference to Figure 3 shows that the above restriction on

Table 1. $g(\theta)$ for $45^\circ \leq \theta \leq 60^\circ$.

θ	$g(\theta)$
45.0°	.0000
46.0°	.0361
47.5°	.0893
50.0°	.1172
52.5°	.1355
55.0°	.1398
57.5°	.1352
60.0°	.1250

Fig. 3. $g(\theta)$ for $45^\circ \leq \theta \leq 60^\circ$.

$g(\theta)$ imposes a corresponding restriction on θ . Thus, the inequality (6.54) requires that,

$$45^\circ < \theta < \theta_c, \quad (6.55)$$

where θ_c is the limiting value of θ for stable infinitesimal orbits. Equation (6.37) shows that the inequality (6.55) imposes a corresponding restriction on ρ . Thus,

$$0 < \rho < \rho_c, \quad (6.56)$$

where ρ_c is the limiting value of ρ for stable infinitesimal orbits.

A linear interpolation of the function $g(\theta)$ to obtain θ_c , followed by its substitution into (6.37), gives the approximate values,

$$\theta_c = 45.397^\circ \quad (6.57)$$

$$\rho_c = 0.0938. \quad (6.58)$$

Equation (6.37) may be approximated by a linear equation by retaining only the linear terms in its Taylor's series expansion. This approximation with (6.30) and (6.36) may be used to obtain an approximate expression for the right side of Equation (6.47). The use of this expression in the stability criterion gives the approximate values,

$$\theta_c = 45.342^\circ \quad (6.59)$$

$$\rho_c = 0.0787, \quad (6.60)$$

which are to be compared with those given by (6.57) and (6.59).

The results of this problem show that libration points exist for all positive values of φ . However, the existence of stable infinitesimal orbits in the neighborhood of a libration point requires that φ be small enough to satisfy the inequality (6.56).

C. Point Mass Centered on an Infinite Rod.

In this problem the potential ϕ is produced by a system consisting of a point mass M located at the origin and an infinitely long, thin rod of linear density h which coincides with the x axis as in Figure 4.

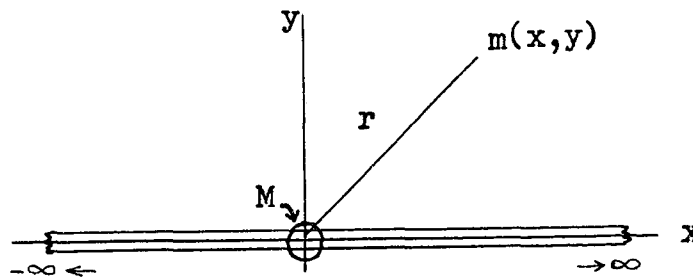


Fig. 4. Mass Centered on Infinite Rod.

Let the ratio of the mass M to the linear density h of the rod be denoted by σ .

$$\sigma = \frac{M}{h} \quad (6.61)$$

If the rod were of length $2a$ the potential ϕ would be

$$\phi = \frac{\gamma M m}{r} + \gamma m h \int_{-a}^a \frac{ds}{\sqrt{(s-x)^2 + y^2}}, \quad (6.62)$$

where γ is the gravitational constant and

$$r^2 = x^2 + y^2. \quad (6.63)$$

A length unit may be chosen such that,

$$\gamma h = \omega^2, \quad (6.64)$$

where ω is the angular velocity of the system. By substituting (6.64) into (6.62), the expression for the potential may be written

$$\phi = \frac{m \omega^2 \sigma}{r} + m \omega^2 \int_{-a}^a \frac{ds}{\sqrt{(x-s)^2 + y^2}} \quad (6.65)$$

The function Ω for the rod of length $2a$ is then

$$\Omega = \frac{r^2}{2} + \frac{\sigma}{r} + \int_{-a}^a \frac{ds}{\sqrt{(x-s)^2 + y^2}} \quad (6.66)$$

For the infinite rod the partial derivatives of are,

$$\Omega_x = x - \sigma \frac{x}{r^3} \quad (6.67)$$

$$\Omega_y = y - \sigma \frac{y}{r^3} - \frac{2}{y} . \quad (6.68)$$

From (6.67) and (6.68) it is evident that for any positive value of σ there exists a pair of libration points at $(0, \pm b)$ in the $x y$ plane, where the positive value of b is obtained from the equation,

$$b - \frac{2}{b} - \frac{\sigma}{b^2} = 0 . \quad (6.69)$$

Since σ is positive, it is clear that b must be greater than $\sqrt{2}$.

The conditions for the existence of stable infinitesimal orbits in the neighborhood of the libration point L_1 at $(0, b)$ are obtained by applying the stability criterion to (6.66). The required partial derivatives of Ω are,

$$\Omega_{xx}(x,y) = 1 - \frac{\sigma}{r^3} + 3 \sigma \frac{x^2}{r^5} \quad (6.70)$$

$$\Omega_{yy}(x,y) = 1 - \frac{\sigma}{r^3} + 3 \sigma \frac{y^2}{r^5} + \frac{2}{y^2} \quad (6.71)$$

$$\Omega_{xy}(x,y) = 3 \sigma \frac{xy}{r^5} . \quad (6.72)$$

When evaluated at the libration point L_1 , the above equations become,

$$\Omega_{xx}(0, b) = \frac{2}{b^2} \quad (6.73)$$

$$\Omega_{yy}(0, b) = 3 - \frac{2}{b^2} \quad (6.74)$$

$$\Omega_{xy}(0, b) = 0. \quad (6.75)$$

The quantities required for the determination of the stability conditions are,

$$\Omega_{xx} + \Omega_{yy} - 4 = -1 \quad (6.76)$$

$$\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 = \frac{2}{b^2} \left(3 - \frac{2}{b^2} \right). \quad (6.77)$$

The condition for stable infinitesimal orbits is then found to be,

$$0 < \frac{24}{b^2} - \frac{16}{b^4} < 1. \quad (6.78)$$

Since b must be greater than $\sqrt{2}$ for σ to be positive, the inequality (6.78) gives,

$$b > 2 + 2\sqrt{2}. \quad (6.79)$$

By applying this restriction of b to Equation (6.69), a corresponding restriction of σ is obtained, namely,

$$\sigma > 52 + 36\sqrt{2}. \quad (6.80)$$

The results show that libration points exist for any positive value of σ . However, the existence of stable infinitesimal orbits in the neighborhood of the libration point requires that σ be large enough to satisfy (6.80).

The stability conditions established in Section V were obtained by retaining only the linear terms in the series expansions of the function Ω_x and Ω_y which appear in the equations of motion. While an orbit about a libration point may be unstable in the infinitesimal sense,

it is possible that it could assume a stable state with finite dimensions. Conversely, while an infinitesimal orbit about a certain libration point may be stable, it is possible that it could become unstable if its dimensions were made finite. A study of these possibilities requires that higher order terms in the Ω_x and Ω_y expressions be retained. In his study of periodic orbits in the restricted problem of three bodies, Pedersen⁶ used terms up to the third order in the Ω_x and Ω_y expressions. A similar procedure will be used here in a further study of the problem of a point mass centered on an infinite rod. Particular attention will be given to that value of σ in whose neighborhood the transition from stable to unstable infinitesimal orbits takes place.

⁶Pedersen, Peder. On the Periodic Orbits in the Neighborhood of the Triangular Equilibrium Points in the Restricted Problem of Three Bodies. Monthly Notices of the Royal Astronomical Society. 94, no. 2: 167-185. December 1933.

VII. HIGHER ORDER APPROXIMATIONS

In this section the analysis of the problem of a point mass centered on an infinite rod is extended by approximating the form of the solution as well as the partial derivatives of Ω required in the equations of motion. The discussion is confined to the study of orbital motion near a libration point. A change of variables is introduced so that the libration point is at the coordinate origin.

The partial derivatives of Ω are approximated by the first ten terms in their respective Taylor's series expansions about the origin. These approximations include the first, second, and third degree terms.

The solutions, assumed to be periodic functions of time, are approximated by retaining the first seven terms in their respective Fourier expansions. The terms retained include those corresponding to the fundamental orbit frequency and its second and third harmonics. The Fourier coefficients are assumed to be such that those which correspond to the fundamental frequency are of the first order, while those corresponding to the second and third harmonics are of second and third order, respectively. The constant terms are assumed to be of the second order.

The first, second, and third degree terms of the expansions of the partial derivatives of Ω are then expressed in terms of the approximate Fourier expressions. The non-linear differential equations of motion are thus approximated by two linear differential equations. A table of coefficients is used to show the resulting correspondence of terms.

The first approximations of the fundamental orbit frequency and the first order Fourier coefficients are obtained from the equations of motion by using only the first order terms. The values thus found are used to obtain the first approximations of the second order Fourier coefficients.

These first approximations of the fundamental orbit frequency and the first and second order coefficients are then used to obtain the second approximations of the first order coefficients. The result of this procedure gives a second approximation of the fundamental orbit frequency.

In the previous section it was shown that, for a certain geometrical configuration, the conditions for infinitesimal orbits in the neighborhood of a libration point change from unstable to stable. This particular case and its neighboring configurations are discussed in the remainder of this section. Results showing the variation of the fundamental orbit frequency with orbit dimensions are

obtained. These results are in agreement with the criterion for stable infinitesimal orbits. The problem is concluded with the definition of a limiting orbit for which the third order Fourier coefficients are calculated and presented with the other coefficients in tabular form.

In the above described problem of a point mass centered on an infinite, rotating rod it was shown that a transition from stable to unstable infinitesimal orbits occurs for a certain critical value of b or σ . Let these critical values of b and σ be denoted by β_0 and σ_0 . Thus,

$$\beta_0 = 2 + 2\sqrt{2} \quad (7.01)$$

$$\sigma_0 = 52 + 36\sqrt{2} \quad (7.02)$$

The discussion presented here will be confined to the orbital motion of the infinitesimal mass m in the neighborhood of the libration point at $(0, b)$ in the $x y$ plane. For simplicity of notation, let,

$$\xi = x \quad (7.03)$$

$$\eta = y - b. \quad (7.04)$$

The libration point under consideration is thus at the origin in the $\xi \eta$ plane. The equations of motion of m are,

$$\ddot{\xi} - 2\dot{\eta} = \Omega_{\xi}(\xi, \eta) \quad (7.05)$$

$$\ddot{\eta} + 2\dot{\xi} = \Omega_{\eta}(\xi, \eta), \quad (7.06)$$

where the dot indicates a derivative with respect to \dot{t} . The quantity \dot{t} is defined by (4.03) and is that choice of time unit required to make the angular velocity of rotation of the coordinate system equal to unity. The partial derivatives of Ω in (7.05) and (7.06) are,

$$\Omega_{\xi}(\xi, \eta) = \xi - \frac{\sigma \xi}{\left[\xi^2 + (\eta + b)^2 \right]^{\frac{3}{2}}} \quad (7.07)$$

$$\Omega_{\eta}(\xi, \eta) = (\eta + b) - \frac{\sigma(\eta + b)}{\left[\xi^2 + (\eta + b)^2 \right]^{\frac{3}{2}}} - \frac{2}{(\eta + b)} \quad (7.08)$$

These partial derivatives will now be approximated by retaining only the first ten terms in their respective series expansions about the libration point (0,0). Each term in these approximations involves a partial derivative of Ω of fourth order or less. However, only seven of these partial derivatives are different from zero at the libration point and consequently, the third order approximations of $\Omega_{\xi}(\xi, \eta)$ and $\Omega_{\eta}(\xi, \eta)$ are of the form,

$$\Omega_{\xi} = C_1 \xi + 2C_2 \xi \eta + C_3 \xi^3 + C_4 \xi \eta^2 \quad (7.09)$$

$$\Omega_{\eta} = C_5 \eta + C_2 \xi^2 + C_6 \eta^2 + C_4 \xi^2 \eta + C_7 \eta^3, \quad (7.10)$$

where the constants C_1 correspond to partial derivatives of Ω , evaluated at the libration point. These constants

are,

$$C_1 = \Omega_{\xi\xi}(0,0) = \frac{2}{b^2} \quad (7.11)$$

$$2C_2 = \Omega_{\xi\xi\eta}(0,0) = \frac{3}{b^2} \left(b - \frac{2}{b} \right) \quad (7.12)$$

$$6C_3 = \Omega_{\xi\xi\xi\xi}(0,0) = \frac{9}{b^3} \left(b - \frac{2}{b} \right) \quad (7.13)$$

$$2C_4 = \Omega_{\xi\xi\eta\eta}(0,0) = \frac{12}{b^3} \left(\frac{2}{b} - b \right) \quad (7.14)$$

$$C_5 = \Omega_{\eta\eta}(0,0) = 3 - \frac{2}{b^2} \quad (7.15)$$

$$2C_6 = \Omega_{\eta\eta\eta}(0,0) = \frac{2}{b} \left(\frac{4}{b^2} - 3 \right) \quad (7.16)$$

$$6C_7 = \Omega_{\eta\eta\eta\eta}(0,0) = \frac{12}{b^2} \left(2 - \frac{3}{b^2} \right) . \quad (7.17)$$

The equations of motion given by (7.05) and (7.06) are thus replaced by their approximations,

$$\ddot{\xi} - 2\dot{\eta} - C_1\xi - 2C_2\xi\eta - C_3\xi^3 - C_4\xi\eta^2 = 0 \quad (7.18)$$

$$\ddot{\eta} + 2\dot{\xi} - C_5\eta - C_2\xi^2 - C_6\eta^2 - C_4\xi^2\eta - C_7\eta^3 = 0. \quad (7.19)$$

To a corresponding degree of approximation it will be assumed that the solutions $\xi(\tau)$ and $\eta(\tau)$ may be represented by retaining only the first seven terms in their respective Fourier expansions. Then,

$$\begin{aligned} \xi = & a_0 + a_1 \cos \omega\tau + a_2 \cos 2\omega\tau + a_3 \cos 3\omega\tau \\ & + a_{-1} \sin \omega\tau + a_{-2} \sin 2\omega\tau + a_{-3} \sin 3\omega\tau \end{aligned} \quad (7.20)$$

$$\begin{aligned} = & b_0 + b_1 \cos \omega\tau + b_2 \cos 2\omega\tau + b_3 \cos 3\omega\tau \\ & + b_{-1} \sin \omega\tau + b_{-2} \sin 2\omega\tau + b_{-3} \sin 3\omega\tau , \end{aligned} \quad (7.21)$$

where ω is the fundamental angular frequency of the orbit.

This new definition of ω is not to be confused with the angular velocity of rotation of the coordinate system which was eliminated from the equations of motion. It is further assumed that the coefficients in (7.20) and (7.21) with subscript 0 are of second order, while all other coefficients with subscript n are of order $|n|$.

The powers, products, and derivatives of ξ and η required in (7.18) and (7.19) may be arranged in tabular form as in Table 2, which is reproduced from Pedersen's work. Only the terms of third order or less are retained.

By using only the first order terms of (7.18), (7.19), (7.20), and (7.21), a relation between the first order coefficients may be obtained. The first order approximations of (7.18) and (7.19) are,

$$\ddot{\xi} - 2\dot{\eta} - C_1\xi = 0 \quad (7.22)$$

$$\ddot{\eta} + 2\dot{\xi} - C_5\eta = 0, \quad (7.23)$$

while the first order approximations of (7.20) and (7.21) are,

$$\xi = a_1 \cos \omega t + a_{-1} \sin \omega t \quad (7.24)$$

$$\eta = b_1 \cos \omega t + b_{-1} \sin \omega t. \quad (7.25)$$

The substitution of (7.24) and (7.25) into (7.22) and (7.23) gives,

$$(\omega^2 + C_1)a_1 + 2\omega b_{-1} = 0 \quad (7.26)$$

$$(\omega^2 + C_1)a_{-1} - 2\omega b_1 = 0 \quad (7.27)$$

$$(\omega^2 + C_5)b_1 - 2\omega a_{-1} = 0 \quad (7.28)$$

Table 2. Scheme of Coefficients

	1	$\cos \omega t$	$\sin \omega t$	$\cos 2\omega t$	$\sin 2\omega t$	$\cos 3\omega t$	$\sin 3\omega t$
ξ η	a_0 b_0	a_1 b_1	a_{-1} b_{-1}	a_2 b_2	a_{-2} b_{-2}	a_3 b_3	a_{-3} b_{-3}
ξ $\dot{\eta}$...	ωa_{-1} ωb_{-1}	$-\omega a_1$ $-\omega b_1$	$2\omega a_{-2}$ $2\omega b_{-2}$	$-2\omega a_2$ $-2\omega b_2$	$3\omega a_{-3}$ $3\omega b_{-3}$	$-3\omega a_3$ $-3\omega b_3$
ξ $\ddot{\eta}$...	$-\omega^2 a_1$ $-\omega^2 b_1$	$-\omega^2 a_{-1}$ $-\omega^2 b_{-1}$	$-4\omega^2 a_2$ $-4\omega^2 b_2$	$-4\omega^2 a_{-2}$ $-4\omega^2 b_{-2}$	$-9\omega^2 a_3$ $-9\omega^2 b_3$	$-9\omega^2 a_{-3}$ $-9\omega^2 b_{-3}$
ξ^2	$\frac{1}{2}a_1^2 + \frac{1}{2}a_{-1}^2$	$2a_0a_1$ $+ a_1a_2$ $+ a_{-1}a_{-2}$	$2a_0a_{-1}$ $+ a_1a_{-2}$ $- a_{-1}a_2$	$\frac{1}{2}a_1^2 - \frac{1}{2}a_{-1}^2$	a_1a_{-1}	$a_1a_2 - a_{-1}a_{-2}$	$a_1a_{-2} + a_{-1}a_2$
$\xi\eta$	$\frac{1}{2}a_1b_1 + \frac{1}{2}a_{-1}b_{-1}$	$\frac{1}{2}(a_1b_2 + a_2b_1)$ $+ \frac{1}{2}(a_{-1}b_{-2} + a_{-2}b_{-1})$ $+ a_1b_0 + a_0b_1$	$\frac{1}{2}(a_1b_{-2} + a_{-2}b_1)$ $- \frac{1}{2}(a_{-1}b_2 + a_2b_{-1})$ $+ a_{-1}b_0 + a_0b_{-1}$	$\frac{1}{2}a_1b_1 - \frac{1}{2}a_{-1}b_{-1}$	$\frac{1}{2}a_1b_{-1} + \frac{1}{2}a_{-1}b_1$	$\frac{1}{2}(a_1b_2 + a_2b_1)$ $- \frac{1}{2}(a_{-1}b_{-2} + a_{-2}b_{-1})$	$\frac{1}{2}(a_1b_{-2} + a_{-2}b_1)$ $+ \frac{1}{2}(a_{-1}b_2 + a_2b_{-1})$
η^2	$\frac{1}{2}b_1^2 + \frac{1}{2}b_{-1}^2$	$2b_0b_1$ $+ b_1b_2$ $+ b_{-1}b_{-2}$	$2b_0b_{-1}$ $+ b_1b_{-2}$ $- b_{-1}b_2$	$\frac{1}{2}b_1^2 - \frac{1}{2}b_{-1}^2$	b_1b_{-1}	$b_1b_2 - b_{-1}b_{-2}$	$b_1b_{-2} + b_{-1}b_2$
ξ^3	...	$\frac{3}{2}a_1^3 + \frac{3}{2}a_1a_{-1}^2$	$\frac{3}{2}a_{-1}^3 + \frac{3}{2}a_1^2a_{-1}$	$\frac{1}{2}a_1^3 - \frac{3}{2}a_1a_{-1}^2$	$-\frac{1}{2}a_{-1}^3 + \frac{3}{2}a_1^2a_{-1}$
$\xi^2\eta$...	$\frac{3}{2}a_1^2b_1$ $+ \frac{1}{2}a_{-1}^2b_1$ $+ \frac{1}{2}a_1a_{-1}b_{-1}$	$\frac{1}{2}a_1^2b_{-1}$ $+ \frac{3}{2}a_{-1}^2b_{-1}$ $+ \frac{1}{2}a_1a_{-1}b_1$	$\frac{1}{2}a_1^2b_1$ $- \frac{1}{2}a_{-1}^2b_1$ $- \frac{1}{2}a_1a_{-1}b_{-1}$	$\frac{1}{2}a_1^2b_{-1}$ $- \frac{1}{2}a_{-1}^2b_{-1}$ $+ \frac{1}{2}a_1a_{-1}b_1$
$\xi\eta^2$...	$\frac{3}{2}a_1b_1^2$ $+ \frac{1}{2}a_1b_{-1}^2$ $+ \frac{1}{2}a_{-1}b_1b_{-1}$	$\frac{1}{2}a_{-1}b_1^2$ $+ \frac{3}{2}a_{-1}b_{-1}^2$ $+ \frac{1}{2}a_1b_1b_{-1}$	$\frac{1}{2}a_1b_1^2$ $- \frac{1}{2}a_1b_{-1}^2$ $- \frac{1}{2}a_{-1}b_1b_{-1}$	$\frac{1}{2}a_{-1}b_1^2$ $- \frac{1}{2}a_{-1}b_{-1}^2$ $+ \frac{1}{2}a_1b_1b_{-1}$
η^3	...	$\frac{3}{2}b_1^3 + \frac{3}{2}b_1b_{-1}^2$	$\frac{3}{2}b_{-1}^3 + \frac{3}{2}b_1^2b_{-1}$	$\frac{1}{2}b_1^3 - \frac{3}{2}b_1b_{-1}^2$	$-\frac{1}{2}b_{-1}^3 + \frac{3}{2}b_1^2b_{-1}$

$$(\omega^2 + C_5)b_{-1} + 2\omega a_1 = 0 . \quad (7.29)$$

Equations (7.26) and (7.28) may be solved to give,

$$b_{-1} = - \frac{\omega^2 + C_1}{2\omega} a_1 \quad (7.30)$$

$$a_{-1} = \frac{\omega^2 + C_5}{2\omega} b_1 . \quad (7.31)$$

The substitution of (7.30) into (7.29) and (7.31) into (7.27) gives,

$$[\omega^4 + (C_1 + C_5 - 4)\omega^2 + C_1C_5] a_1 = 0 \quad (7.32)$$

$$[\omega^4 + (C_1 + C_5 - 4)\omega^2 + C_1C_5] b_1 = 0 . \quad (7.33)$$

Equations (7.32 and (7.33) have non-trivial solutions, if,

$$[\omega^4 + (C_1 + C_5 - 4)\omega^2 + C_1C_5] = 0 , \quad (7.34)$$

or if,

$$2\omega^2 = -(C_1 + C_5 - 4) \pm \sqrt{(C_1 + C_5 - 4)^2 - 4C_1C_5} . \quad (7.35)$$

From (7.35) it is evident that the conditions for to be real and different from zero are,

$$C_1 + C_5 - 4 < 0 \quad (7.36)$$

$$C_1C_5 > 0 \quad (7.37)$$

$$4C_1C_5 < (C_1 + C_5 - 4)^2 . \quad (7.38)$$

Since $\Omega_{\xi\eta}(0,0)$ is zero it can be seen from the definitions of C_1 and C_5 that the inequalities represented by (7.36), (7.37), and (7.38) are the same as the conditions required for stable infinitesimal orbits in the neighborhood of the libration point at (0,0) in the $\xi\eta$ plane. It was shown for this problem that these conditions are satisfied if,

$$\sigma > \sigma_0, \quad (7.39)$$

where σ_0 is given by (7.02). It will be assumed that (7.39) is satisfied and that the first approximation of ω is given by (7.35).

From the above first order analysis it is evident that a_1 and b_1 are arbitrary constants from which a_{-1} and b_{-1} can be determined. From (7.25) it can be seen that by a proper choice of the time origin, the coefficient b_1 can be made to vanish. It then follows from (7.31) that a_{-1} must also vanish. By denoting a_1 by ϵ , the first approximations of the first order coefficients may be written,

$$a_1 = \epsilon \quad (7.40)$$

$$b_{-1} = -\frac{\omega^2 + C_1}{2\omega} \epsilon \quad (7.41)$$

$$a_{-1} = 0 \quad (7.42)$$

$$b_1 = 0, \quad (7.43)$$

where ω is given by (7.35). Only the positive value of ω will be used in this discussion.

The values given by (7.40), (7.41), (7.42), and (7.43) may be used to evaluate the second order terms in the first column corresponding to ξ^2 , $\xi\eta$, and η^2 in Table 2. These terms are then,

$$\frac{1}{2} a_1^2 + a_{-1}^2 = \frac{1}{2} \epsilon^2 \quad (7.44)$$

$$\frac{1}{2} a_1 b_1 + \frac{1}{2} a_{-1} b_{-1} = 0 \quad (7.45)$$

$$\frac{1}{2}b_1^2 + \frac{1}{2}b_{-1}^2 = \frac{(\omega^2 + C_1)^2}{8\omega^2} \epsilon^2 . \quad (7.46)$$

The coefficients a_0 and b_0 may be obtained by equating to zero the constant terms which appear in the left members of (7.18) and (7.19) after substituting (7.20) and (7.21) into (7.18) and (7.19) and making use of (7.44), (7.45), and (7.46). This procedure gives,

$$a_0 = 0 \quad (7.47)$$

$$b_0 = - \frac{C_6\omega^4 + (4C_2 + 2C_1C_6)\omega^2 + C_6C_1^2}{8C_5\omega^2} \epsilon^2 . \quad (7.48)$$

The values given by (7.40), (7.41), (7.42), and (7.43) may also be used to evaluate the second order terms in Table 2 which correspond to $\cos 2\omega t$ and $\sin 2\omega t$ and ξ^2 , η^2 , and ζ^2 . These terms then become,

$$\frac{1}{2}a_1^2 - \frac{1}{2}a_{-1}^2 = \frac{1}{2}\epsilon^2 \quad (7.49)$$

$$\frac{1}{2}a_1b_1 - \frac{1}{2}a_{-1}b_{-1} = 0 \quad (7.50)$$

$$\frac{1}{2}b_1^2 - \frac{1}{2}b_{-1}^2 = - \frac{(\omega^2 + C_1)^2}{8\omega^2} \epsilon^2 \quad (7.51)$$

$$a_1a_{-1} = 0 \quad (7.52)$$

$$\frac{1}{2}a_1b_{-1} + \frac{1}{2}a_{-1}b_1 = - \frac{(\omega^2 + C_1)}{4\omega} \epsilon^2 \quad (7.53)$$

$$b_1b_{-1} = 0 . \quad (7.54)$$

The above equations are used to obtain the coefficients a_2 , a_{-2} , b_2 , and b_{-2} after substituting (7.20) and (7.21) into (7.18) and (7.19) and equating to zero the coefficients of $\cos 2\omega\tau$ and $\sin 2\omega\tau$. The coefficients thus found are,

$$a_2 = 0 \quad (7.55)$$

$$a_{-2} = \frac{(C_6 + 4C_2)\omega^4 + (4C_1C_2 + C_2C_5 + 2C_1C_6 - 4C_2)\omega^2 + C_1^2C_6 + C_1C_2C_5}{6\omega^3 [5\omega^2 + (C_1 + C_5 - 4)]} \epsilon^2 \quad (7.56)$$

$$b_2 = \frac{4C_6\omega^6 + 9C_1C_6\omega^4 + 6C_1(C_1C_6 + 2C_2)\omega^2 + C_1^3C_6}{24\omega^4 [5\omega^2 + (C_1 + C_5 - 4)]} \epsilon^2 \quad (7.57)$$

$$b_{-2} = 0 \quad (7.58)$$

In the foregoing discussion the first approximations of the coefficients of the first and second order are given in terms of the constants C_1 . These constants, in turn, are defined by the partial derivatives of $\Omega(\xi, \eta)$ at $(0, 0)$. Since the five coefficients a_0 , a_{-1} , b_1 , a_2 and b_{-2} are zero, the coefficient scheme given by Table 2 is replaced by that in Table 3.

From Equations (7.40), (7.41), (7.48), (7.56), and (7.57), which define a_1 , b_{-1} , b_0 , a_{-2} , and b_2 , it is evident that the third order terms in Table 3 corresponding to $\cos \omega\tau$ and $\sin \omega\tau$ and ξ^2 , $\xi\eta$, η^2 , ξ^3 , $\xi^2\eta$, $\xi\eta^2$, and η^3 each contain ϵ^3 as a factor. Since ϵ is considered

Table 3. Condensed Scheme of Coefficients.

	1	$\cos \omega t$	$\sin \omega t$	$\cos 2\omega t$
ξ	...	a_1
η	b_0	...	b_{-1}	b_2
$\dot{\xi}$	$-\omega a_1$	$2\omega a_{-2}$
$\dot{\eta}$...	ωb_{-1}
$\ddot{\xi}$...	$-\omega^2 a_1$
$\ddot{\eta}$	$-\omega^2 b_{-1}$	$-4\omega^2 a_{-2}$
ξ^2	$\frac{1}{2} a_1^2$...	$a_1 a_{-2}$	$\frac{1}{2} a_1^2$
$\xi \eta$...	$\frac{1}{2}(a_1 b_2 + a_{-2} b_{-1}) + a_1 b_0$
η^2	$\frac{1}{2} b_{-1}^2$...	$2b_0 b_{-1} - b_{-1} b_2$	$-\frac{1}{2} b_{-1}^2$
ξ^3	...	$\frac{3}{4} a_1^3$
$\xi^2 \eta$	$\frac{1}{4} a_1^2 b_{-1}$...
$\xi \eta^2$...	$\frac{1}{4} a_1^2 b_{-1}$
η^3	$\frac{3}{4} b_{-1}^3$...

is.

$\sin \omega t$	$\cos 2\omega t$	$\sin 2\omega t$	$\cos 3\omega t$	$\sin 3\omega t$
...	...	a_{-2}	a_3	a_{-3}
b_{-1}	b_2	...	b_3	b_{-3}
$-\omega a_1$	$2\omega a_{-2}$...	$3\omega a_{-3}$	$-3\omega a_3$
...	...	$-2\omega b_2$	$3\omega b_{-3}$	$-3\omega b_3$
...	...	$-4\omega^2 a_{-2}$	$-9\omega^2 a_3$	$-9\omega^2 a_{-3}$
$-\omega^2 b_{-1}$	$-4\omega^2 b_{-2}$...	$-9\omega^2 b_3$	$-9\omega^2 b_{-3}$
$a_1 a_{-2}$	$\frac{1}{2} a_1^2$	$a_1 a_{-2}$
...	...	$\frac{1}{2} a_1 b_{-1}$	$\frac{1}{2}(a_1 b_2 - a_{-2} b_{-1})$...
$2b_0 b_{-1} - b_{-1} b_2$	$-\frac{1}{2} b_{-1}^2$	$b_{-1} b_2$
...	$\frac{1}{4} a_1^3$...
$\frac{1}{4} a_1^2 b_{-1}$	$\frac{1}{4} a_1^2 b_{-1}$
...	$-\frac{1}{4} a_1 b_{-1}^2$...
$\frac{3}{4} b_{-1}^3$	$-\frac{1}{4} b_{-1}^3$

small, these third order terms may be treated as constants determined by their first approximations to obtain second approximations of a_1 and b_{-1} . The refined value of a_1 will be used to establish a relation between ω and ϵ . The use of (7.18) and (7.19) and the columns of Table 3 which correspond to $\cos \omega \tau$ and $\sin \omega \tau$ gives,

$$(\omega^2 + C_1)a_1 + 2\omega b_{-1} + p_1 = 0 \quad (7.59)$$

$$2\omega a_1 + (\omega^2 + C_5)b_{-1} + p_2 = 0, \quad (7.60)$$

where,

$$p_1 = C_2(a_1 b_2 + a_{-2} b_{-1} + 2a_1 b_0) + \frac{3C_3}{4} a_1^3 + \frac{C_4}{4} a_1 b_{-1}^2 \quad (7.61)$$

$$p_2 = C_2 a_1 a_{-2} + C_6(2b_0 b_{-1} - b_{-1} b_2) + \frac{C_4}{4} a_1^2 b_{-1} + \frac{3C_7}{4} b_{-1}^3. \quad (7.62)$$

The quantities p_1 and p_2 are to be evaluated by using the first approximations of a_1 , b_{-1} , b_0 , a_{-2} , and b_2 given by (7.40), (7.41), (7.48), (7.56), and (7.57). Equations (7.59) and (7.60) may be solved for a_1 to give,

$$a_1 = \frac{2\omega p_2 - (\omega^2 + C_5)p_1}{(\omega^2 + C_1)(\omega^2 + C_5) - 4\omega^2}. \quad (7.63)$$

By setting a_1 in (7.63) equal to ϵ , a new equation for a refined value of ω is obtained.

Thus,

$$\omega^4 + (C_1 + C_5 - 4)\omega^2 + C_1 C_5 + [(\omega^2 + C_5)q_1 - 2\omega q_2] = 0, \quad (7.64)$$

where q_1 and q_2 are defined by the equations,

$$\epsilon q_1 = p_1 \quad (7.65)$$

$$\epsilon q_2 = p_2 . \quad (7.66)$$

The discussion is now limited to the case where σ lies in the neighborhood of σ_0 defined by (7.02). Then from (6.69), it is seen that b must lie in the neighborhood of the value β_0 defined by (7.01). Thus σ and b may be written,

$$\sigma = \sigma_0 + \delta \quad (7.67)$$

$$b = \beta_0(1 + \alpha) , \quad (7.68)$$

where α and δ are small compared to unity. The values given by (7.67) and (7.68) may be substituted into (6.69) to yield the relation,

$$\delta = 4(41 + 29\sqrt{2})\alpha . \quad (7.69)$$

The first order relation for ω given by (7.35) shows that for b equal to β_0 the angular frequency ω takes a corresponding value ω_0 , where,

$$\omega_0^2 = \frac{1}{2} . \quad (7.70)$$

As previously mentioned, only positive values of ω will be used in this discussion.

Because of their higher order, the terms enclosed by the brackets in (7.64) will be computed by using the values σ_0 , β_0 , and ω_0 for σ , b , and ω . The quantities C_1 when evaluated for b equal to β_0 are,

$$C_1 = \frac{3-2\sqrt{2}}{2} \quad (7.71)$$

$$C_2 = \frac{3}{8} (5-3\sqrt{2}) \quad (7.72)$$

$$c_3 = \frac{3}{16} (8\sqrt{2} - 11) \quad (7.73)$$

$$c_4 = \frac{3}{4} (11 - 8\sqrt{2}) \quad (7.74)$$

$$c_5 = \frac{3+2\sqrt{2}}{2} \quad (7.75)$$

$$c_6 = \sqrt{2} - 2 \quad (7.76)$$

$$c_7 = \frac{20\sqrt{2} - 27}{8} . \quad (7.77)$$

These equations and (7.70) are used to evaluate the quantities a_1 , b_{-1} , b_0 , a_{-2} , and b_2 given by (7.40), (7.41), (7.48), (7.56), and (7.57). After performing the necessary calculations these quantities are found to be,

$$a_1 = \epsilon \quad (7.78)$$

$$b_{-1} = (1 - \sqrt{2}) \epsilon \quad (7.79)$$

$$b_0 = \frac{383 - 271\sqrt{2}}{8} \epsilon^2 \quad (7.80)$$

$$a_{-2} = \frac{125 - 88\sqrt{2}}{36} \epsilon^2 \quad (7.81)$$

$$b_2 = \frac{381\sqrt{2} - 541}{72} \epsilon^2 \quad (7.82)$$

The quantities p_1 and p_2 defined by (7.61) and (7.62) may now be evaluated. The results are,

$$p_1 = \frac{49,767 - 35,190\sqrt{2}}{144} \in \mathbb{R} \quad (7.83)$$

$$p_2 = \frac{88,659\sqrt{2} - 125,381}{144} \in \mathbb{R} . \quad (7.84)$$

The quantity Q enclosed by the brackets in (7.64) is evaluated by using (7.65), (7.66), (7.70), (7.75), (7.83), and (7.84). The result is found to be,

$$Q = k\epsilon^2, \quad (7.85)$$

where,

$$k = \frac{97\sqrt{2} - 82}{36(338 + 239\sqrt{2})} . \quad (7.86)$$

The substitution of (7.01) and (7.68) into (7.11) and (7.15) gives,

$$C_1 = \frac{3-2\sqrt{2}}{2} - (3-2\sqrt{2})\alpha \quad (7.87)$$

$$C_5 = \frac{3+2\sqrt{2}}{2} + (3-2\sqrt{2})\alpha . \quad (7.88)$$

The product $C_1 C_5$ required in (7.64) is obtained by using (7.87) and (7.88) and is found to be,

$$C_1 C_5 = \frac{1}{4} - 2(3\sqrt{2} - 4)\alpha . \quad (7.89)$$

From (7.11) and (7.15) it is seen that the coefficient of ω^2 in (7.64) is equal to -1 for all values of b concerned. Thus, by using (7.11), (7.15), (7.85), and (7.89), equation (7.64) may be written,

$$\omega^4 - \omega^2 + \frac{1}{4} - 2(3\sqrt{2} - 4)\alpha + k\epsilon^2 = 0, \quad (7.90)$$

or,

$$2\omega^2 = 1 \pm 2\sqrt{2}(3\sqrt{2} - 4)\alpha - k\epsilon^2. \quad (7.91)$$

Necessary and sufficient conditions for ω to be real are,

$$0 \leq 2(3\sqrt{2} - 4)\alpha - k\epsilon^2 \leq \frac{1}{4}. \quad (7.92)$$

The condition given by (7.92) may be written in terms of δ by using (7.69) and (7.86). The inequality (7.92) then becomes,

$$\frac{97\sqrt{2} - 82}{36}\epsilon^2 \leq \delta \leq \frac{97\sqrt{2} - 82}{36}\epsilon^2 + \frac{239\sqrt{2} + 338}{4}. \quad (7.93)$$

Since ϵ and δ have been assumed to be small quantities, it is clear that the term on the right of the above inequality is so large that the upper limit of δ need not be considered. Thus, for real values of ω the orbit dimension ϵ must be restricted so that,

$$\epsilon^2 \leq k_1 \delta, \quad (7.94)$$

where,

$$k_1 = \frac{36}{97\sqrt{2} - 82}. \quad (7.95)$$

From (7.94) it is seen that the dimension ϵ of the orbit must vanish as δ approaches zero. This is in agreement with the stability criterion established for infinitesimal orbits.

By using (7.69), (7.86), and (7.95), the relation between ω and ϵ , given by (7.90), may be written,

$$\left(\omega^2 - \frac{1}{2}\right)^2 + k\epsilon^2 = k k_1 \delta. \quad (7.96)$$

It is evident from (7.94) or (7.96) that the quantity cannot be negative. Equation (7.96) is the equation of an ellipse if $\left(\omega^2 - \frac{1}{2}\right)$ is plotted in terms of ϵ . This is in contrast to the hyperbola equation obtained by Pedersen in his work on the restricted problem of three bodies. Equation (7.96) is illustrated by the sketch in Figure 5 for two different values of δ .

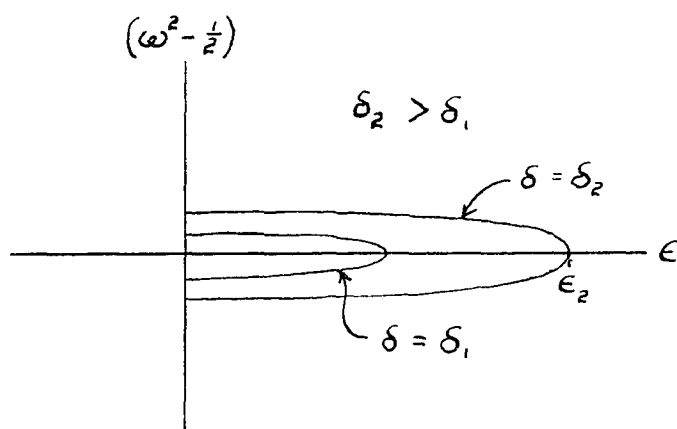


Fig. 5. Sketch of Equation (7.96).

From Figure 5 it is evident that for a given δ_2 the quantity $\left(\omega^2 - \frac{1}{2}\right)$ has two values for all $0 \leq \epsilon < \epsilon_2$.

These two values approach each other as ϵ approaches ϵ_2 . When $\epsilon = \epsilon_2$ the two values of $(\omega^2 - \frac{1}{2})$ become equal to zero. The orbit corresponding to this condition was called the limiting orbit by Pedersen. The value of ω for the limiting orbit is $\frac{\sqrt{2}}{2}$. The discussion will be concluded by calculating the remaining coefficients a_3 , b_3 , a_{-3} , and b_{-3} for a limiting orbit.

Equations for obtaining the first approximations of the coefficients a_3 , b_3 , a_{-3} , and b_{-3} may be obtained from Equations (7.18), (7.19), (7.20), and (7.21) and Table 3 in the manner previously used to obtain other coefficients. The resulting equations are,

$$(C_1 + 9\omega^2)a_{-3} - 6\omega b_3 = 0 \quad (7.97)$$

$$-6\omega a_{-3} + (C_5 + 9\omega^2)b_3 = 0 \quad (7.98)$$

$$(C_1 + 9\omega^2)a_3 + 6\omega b_{-3} = s_1 \quad (7.99)$$

$$6\omega a_3 + (C_5 + 9\omega^2)b_{-3} = s_2, \quad (7.100)$$

where,

$$s_1 = \frac{C_4}{4} a_1 b_{-1}^2 - \frac{C_3}{4} a_1^3 - C_2(a_1 b_2 - a_{-2} b_{-1}) \quad (7.101)$$

$$s_2 = \frac{C_7}{4} b_{-1}^3 - \frac{C_4}{4} a_1^2 b_{-1} - C_6 b_{-1} b_2 - C_2 a_1 a_{-2}. \quad (7.102)$$

Because of their order and their intended use in the equations for a limiting orbit, the coefficients a_3 , b_3 , a_{-3} , and b_{-3} in the above equations may be evaluated by using σ_0 , β_0 , and ω_0 in place of σ , b , and ω . Under these

assumptions the first approximations of the coefficients a_3 , b_3 , a_{-3} , and b_{-3} are found to be,

$$a_3 = \frac{1818 - 1285\sqrt{2}}{384} \epsilon^3 \quad (7.103)$$

$$b_3 = 0 \quad (7.104)$$

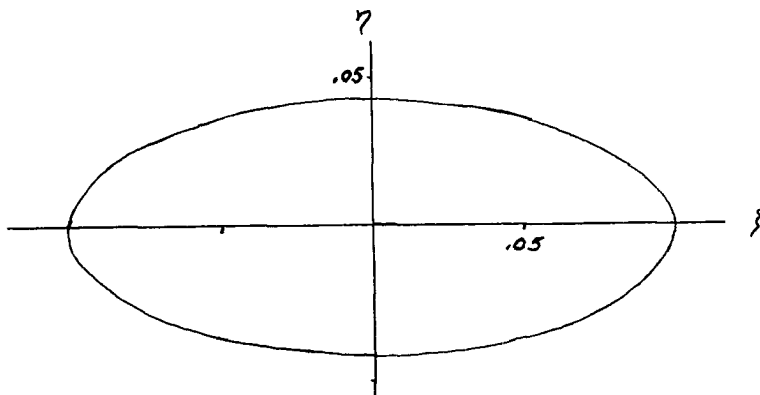
$$a_{-3} = 0 \quad (7.105)$$

$$b_{-3} = \frac{19.735\sqrt{2} - 27.912}{1152} \epsilon^3 \quad (7.106)$$

The values found for the coefficients in (7.20) and (7.21) are assembled in tabular form in Table 4. From these values a limiting orbit for the value of ϵ associated with each particular value of δ may be plotted as in Figure 6, where ϵ is taken as 0.1. Since ϵ is small, the shape of the orbit is nearly determined by the terms a_1 and b_{-1} .

Table 4. Limiting Orbit Coefficients

$a_0 = 0$	$b_0 = \frac{383-271\sqrt{2}}{8} \epsilon^2$
$a_1 = \epsilon$	$b_1 = 0$
$a_{-1} = 0$	$b_{-1} = (1 - \sqrt{2})\epsilon$
$a_2 = 0$	$b_2 = \frac{381\sqrt{2} - 541}{72} \epsilon^2$
$a_{-2} = \frac{125 - 88\sqrt{2}}{36} \epsilon^2$	$b_{-2} = 0$
$a_3 = \frac{1818 - 1285\sqrt{2}}{384} \epsilon^3$	$b_3 = 0$
$a_{-3} = 0$	$b_{-3} = \frac{19,735\sqrt{2} - 27,912}{1152} \epsilon^3$

Fig. 6. Limiting Orbit for $\epsilon = 0.1$.

VIII. SUMMARY

The paper is concerned with the motion of a point mass m located in a space referred to an xyz-rectangular coordinate system which rotates about the z-axis with a constant angular velocity ω . In addition to the centrifugal and Coriolis forces associated with rotation, the point mass m is assumed to be acted upon by a force derivable from a particular potential function $\phi(x, y, z)$.

Matrix notation is used to derive the equations of motion of m by a transformation of Newton's equations of motion. The equations of motion thus obtained are

$$m \left(\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - \omega^2 x \right) = \frac{\partial \phi}{\partial x} \quad (8.01)$$

$$m \left(\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} - \omega^2 y \right) = \frac{\partial \phi}{\partial y} \quad (8.02)$$

$$m \frac{d^2 z}{dt^2} = \frac{\partial \phi}{\partial z} \quad (8.03)$$

These differential equations are non-linear unless the potential function ϕ belongs to a restricted class of functions.

A particular potential function yielding a set of linear differential equations is illustrated by a problem in which the point mass m is connected to n fixed anchor points (x_i, y_i, z_i) by means of n linear springs, each of a particular stiffness k_i , ($i = 1, 2, 3, \dots, n$). It is shown that

the resulting motion of m is the same as that which would occur if the entire system of springs and anchor points were replaced by an equivalent single spring attached to an equivalent single anchor point. Explicit solutions are obtained by use of the Laplace transformation.

Following this linear problem, the remainder of the paper concerns the motion in the xy -plane of the point mass m when the potential ϕ is that due to certain distributions of Newtonian potential sources in the xy -plane. The substitutions $\tau = \omega t$, $r^2 = x^2 + y^2$, and

$$\Omega = \frac{r^2}{2} + \frac{\phi}{m\omega^2} , \quad (8.04)$$

applied to Equations (8.01) and (8.02) yield

$$\ddot{x} - 2\dot{y} = \Omega_x \quad (8.05)$$

$$\ddot{y} + 2\dot{x} = \Omega_y . \quad (8.06)$$

The dot indicates differentiation with respect to τ and the subscript on Ω denotes partial differentiation with respect to the variable indicated. Points in the xy -plane where $\Omega_x = \Omega_y = 0$ are known as libration points.

Due to the non-linear nature of the differential equations associated with the gravitational potential, the scope of the subsequent discussion is limited to the study of the motion of the point mass m in the neighborhood of a libration point. When placed at a libration point L , the point mass m may be considered to be in a state of equilibrium, since it will remain at L in the absence of

disturbances from sources external to the system. The stability of this state of equilibrium is determined by the existence of stable infinitesimal orbits in the neighborhood of the libration point. The conditions for the existence of stable infinitesimal orbits about a libration point are obtained by approximating (8.05) and (8.06) by retaining only the linear terms in the series expansions about the libration point of Ω_x and Ω_y . Necessary and sufficient conditions for stable infinitesimal orbits are shown to be

$$\Omega_{xx} + \Omega_{yy} - 4 < 0 \quad (8.07)$$

$$\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2 > 0 \quad (8.08)$$

$$4(\Omega_{xx}\Omega_{yy} - \Omega_{xy}^2) < (\Omega_{xx} + \Omega_{yy} - 4)^2, \quad (8.09)$$

where the partial derivatives Ω_{xx} , Ω_{yy} , and Ω_{xy} are evaluated at the libration point.

Under a continuous variation of the geometry of a potential-producing system, the above stability conditions may change. This behavior is illustrated in three different systems which are examined with regard to the stability criterion. The first of these is the well-known restricted problem of three bodies, the second is a restricted problem of four bodies and the third is a problem of a point mass centered on a rotating, infinite rod.

While an orbit about a libration point may be unstable in the infinitesimal sense, it is possible that it could

assume a stable state with finite dimensions. Conversely, while an infinitesimal orbit about a certain libration point may be stable, it is possible that it could become unstable if its dimensions were made finite. A study of these possibilities requires that higher order terms in the Ω_x and Ω_y expansions be retained.

The paper is concluded with a more extended study of the problem of a point mass centered on a rotating, infinite rod, following a method previously used by Pedersen⁷ in connection with the restricted problem of three bodies. The partial derivatives of Ω are approximated by the first ten terms in their respective Taylor's series expansions about the libration point. The solutions, assumed to be periodic functions of time, are approximated by retaining the first seven terms in their respective Fourier expansions. The terms retained include those corresponding to the fundamental orbit frequency and its second and third harmonics. The Fourier coefficients are assumed to be such that those terms which correspond to the fundamental frequency are of the first order, while those corresponding to the second and third harmonics are of the second and third order, respectively. The constant terms are assumed to be of the second order. Results showing the variation of the

⁷Pedersen, op. cit., p. 42

fundamental orbit frequency with orbit dimensions are obtained. The problem is concluded with the definition of a limiting orbit for which the Fourier coefficients are presented in tabular form.

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X. ACKNOWLEDGMENT

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XI. APPENDICES

A. The Solution of Equations (3.17) and (3.18).

With $D(s)$ defined by (3.20), Equations (3.17) and (3.18) give,

$$\begin{aligned} [x(s)][D(s)] &= \left(\frac{a_1}{s} + \dot{x}_0 - 2\omega y_0 + s x_0 \right) (s^2 + c^2) \\ &\quad + 2\omega s \left(\frac{a_2}{s} + \dot{y}_0 + 2\omega x_0 + s y_0 \right). \end{aligned}$$

By collecting terms, and using (3.19), one obtains

$$[x(s)][D(s)] = A_3 s^3 + A_2 s^2 + A_1 s + A_0 + A_{-1} s^{-1},$$

where,

$$A_3 = x_0$$

$$A_2 = \dot{x}_0$$

$$A_1 = a_1 + 2\omega \dot{y}_0 + (b^2 + 3\omega^2)x_0$$

$$A_0 = (b^2 - \omega^2)(\dot{x}_0 - 2\omega y_0) + 2\omega a_2$$

$$A_{-1} = a_1(b^2 - \omega^2).$$

Then, from (3.22) it is seen that,

$$x(s) = \frac{A_3 s^3 + A_2 s^2 + A_1 s + A_0 + A_{-1} s^{-1}}{[s^2 + (b + \omega)^2][s^2 + (b - \omega)^2]}.$$

With $D(s)$ defined by (3.20), Equations (3.17) and (3.18) also give,

$$[y(s)][D(s)] = \left(\frac{a_2}{s} + \dot{y}_0 + 2\omega x_0 + s y_0 \right) (s^2 + c^2)$$

$$-2\omega s \left(\frac{a_1}{s} + \dot{x}_0 - 2\omega y_0 + s x_0 \right) .$$

By collecting terms and using (3.19), one obtains

$$[y(s)] [D(s)] = B_3 s^3 + B_2 s^2 + B_1 s + B_0 + B_{-1} s^{-1} ,$$

where,

$$B_3 = y_0$$

$$B_2 = \dot{y}_0$$

$$B_1 = a_2 - 2\omega \dot{x}_0 + (b^2 + 3\omega^2) y_0$$

$$B_0 = (b^2 - \omega^2)(\dot{y}_0 + 2\omega x_0) - 2\omega a_1$$

$$B_{-1} = a_2(b^2 - \omega^2) .$$

Then, from (3.22) it is seen that,

$$y(s) = \frac{B_3 s^3 + B_2 s^2 + B_1 s + B_0 + B_{-1} s^{-1}}{[s^2 + (b + \omega)^2][s^2 + (b - \omega)^2]} .$$

B. The Inversion of Equations (3.23) and (3.24).

Equations (3.23) and (3.24) may be written in the form,

$$x(s) = [A_3 s^3 + A_2 s^2 + A_1 s + A_0 + A_{-1} s^{-1}] f(s)$$

$$y(s) = [B_3 s^3 + B_2 s^2 + B_1 s + B_0 + B_{-1} s^{-1}] f(s) ,$$

where,

$$f(s) = \frac{1}{[s^2 + (b + \omega)^2][s^2 + (b - \omega)^2]} .$$

The inverse $L^{-1} \{ f(s) \}$ of $f(s)$ is,

$$L^{-1}\{f(s)\} = G(t) ,$$

where,

$$G(t) = F_1(t) , \quad \text{if } b^2 \neq \omega^2 ,$$

and,

$$G(t) = F_2(t) , \quad \text{if } b^2 = \omega^2 .$$

The functions $F_1(t)$ and $F_2(t)$ are,

$$F_1(t) = \frac{(b+\omega)\sin(b-\omega)t - (b-\omega)\sin(b+\omega)t}{4b\omega(b^2-\omega^2)}$$

$$F_2(t) = \frac{2\omega t - \sin 2\omega t}{8\omega^3} .$$

From the equation,

$$F_1(t) = \frac{(b+\omega)\sin(b-\omega)t - (b-\omega)\sin(b+\omega)t}{4b\omega(b^2-\omega^2)} ,$$

the derivatives of $F_1(t)$ are seen to be,

$$F_1'(t) = \frac{1}{4b\omega} [\cos(b-\omega)t - \cos(b+\omega)t]$$

$$F_1''(t) = \frac{(b+\omega)}{4b\omega} \sin(b+\omega)t - \frac{(b-\omega)}{4b\omega} \sin(b-\omega)t$$

$$F_1'''(t) = \frac{(b+\omega)^2}{4b\omega} \cos(b+\omega)t - \frac{(b-\omega)^2}{4b\omega} \cos(b-\omega)t .$$

The integral of $F_1(t)$ is,

$$\int_0^t F(t) dt = \int_0^t \frac{\sin(b-\omega)\tau}{4b\omega(b-\omega)} d\tau - \int_0^t \frac{\sin(b+\omega)\tau}{4b\omega(b+\omega)} d\tau$$

$$\begin{aligned}
&= \left[\frac{\cos(b+\omega)\tau}{4b\omega(b+\omega)^2} \right]_0^t - \left[\frac{\cos(b-\omega)\tau}{4b\omega(b-\omega)^2} \right]_0^t \\
&= \frac{\cos(b+\omega)t}{4b\omega(b+\omega)^2} - \frac{\cos(b-\omega)t}{4b\omega(b-\omega)^2} + \frac{1}{(b^2-\omega^2)^2} .
\end{aligned}$$

From the above definitions of the function $F_1(t)$ and its derivatives it is evident that,

$$F_1(0) = F_1'(0) = F_1''(0) = 0 .$$

From the equation,

$$F_2(t) = \frac{2\omega t - \sin 2\omega t}{8\omega^3} ,$$

the derivatives of $F_2(t)$ are seen to be,

$$F_2'(t) = \frac{1 - \cos 2\omega t}{4\omega^2}$$

$$F_2''(t) = \frac{\sin 2\omega t}{2\omega}$$

$$F_2'''(t) = \cos 2\omega t .$$

The integral of $F_2(t)$ is,

$$\begin{aligned}
\int_0^t F_2(t) dt &= \int_0^t \frac{2\omega\tau - \sin 2\omega\tau}{8\omega^3} d\tau \\
&= \left[\frac{2\omega^2 \tau^2 + \cos 2\omega\tau}{16\omega^4} \right]_0^t = \frac{2\omega^2 t^2 + \cos 2\omega t - 1}{16\omega^4} .
\end{aligned}$$

From the above definitions of the function $F_2(t)$ and its

derivatives it is seen that,

$$F_2(0) = F_2'(0) = F_2''(0) = 0 .$$

The definition of the functions $G(t)$, $F_1(t)$, and $F_2(t)$ shows that,

$$G(0) = G'(0) = G''(0) = 0 .$$

Since the initial values of the function $G(t)$ and its derivatives vanish, the properties of the Laplace transform given by (3.38), (3.39), (3.40), and (3.41) may be used in the expressions for $x(s)$ and $y(s)$ to give,

$$x(t) = A_3 G'''(t) + A_2 G''(t) + A_1 G'(t) + A_0 G(t) + A_{-1} \int_0^t G(\tau) d\tau$$

$$y(t) = B_3 G'''(t) + B_2 G''(t) + B_1 G'(t) + B_0 G(t) + B_{-1} \int_0^t G(\tau) d\tau ,$$

where,

$$\begin{array}{ll} G(t) = F_1(t) & \text{if } b^2 \neq \omega^2 \\ \text{and} & \\ G(t) = F_2(t) & \text{if } b^2 = \omega^2 . \end{array}$$

Thus, the inversions of $x(s)$ and $y(s)$ are of one form if $b^2 \neq \omega^2$, and of another form in the case where $b^2 = \omega^2$.

If $b^2 \neq \omega^2$, then,

$$x(t) = A_3 \left[\frac{(b+\omega)^2}{4b\omega} \cos(b+\omega)t - \frac{(b-\omega)^2}{4b\omega} \cos(b-\omega)t \right]$$

$$+ A_2 \left[\frac{(b+\omega)}{4b\omega} \sin(b+\omega)t - \frac{(b-\omega)}{4b\omega} \sin(b-\omega)t \right]$$

$$\begin{aligned}
& + A_1 \left[\frac{\cos(b-\omega)t}{4b\omega} - \frac{\cos(b+\omega)t}{4b\omega} \right] \\
& + A_0 \left[\frac{\sin(b-\omega)t}{4b\omega(b-\omega)} - \frac{\sin(b+\omega)t}{4b\omega(b+\omega)} \right] \\
& + A_{-1} \left[\frac{\cos(b+\omega)t}{4b\omega(b+\omega)^2} - \frac{\cos(b-\omega)t}{4b\omega(b-\omega)^2} + \frac{1}{(b^2-\omega^2)^2} \right], \\
y(t) = & B_3 \left[\frac{(b+\omega)}{4b\omega} \cos(b+\omega)t - \frac{(b-\omega)^2}{4b\omega} \cos(b-\omega)t \right] \\
& + B_2 \left[\frac{(b+\omega)}{4b\omega} \sin(b+\omega)t - \frac{(b-\omega)}{4b\omega} \sin(b-\omega)t \right] \\
& + B_1 \left[\frac{\cos(b-\omega)t}{4b\omega} - \frac{\cos(b+\omega)t}{4b\omega} \right] \\
& + B_0 \left[\frac{\sin(b-\omega)t}{4b\omega(b-\omega)} - \frac{\sin(b+\omega)t}{4b\omega(b+\omega)} \right] \\
& + B_{-1} \left[\frac{\cos(b+\omega)t}{4b\omega(b+\omega)^2} - \frac{\cos(b-\omega)t}{4b\omega(b-\omega)^2} + \frac{1}{(b^2-\omega^2)^2} \right].
\end{aligned}$$

By collecting terms and using the definitions of the coefficients A_1 and B_1 , the last two equations may be written,

$$\begin{aligned}
x(t) = & C_1 + C_2 \sin(b+\omega)t + C_3 \cos(b+\omega)t \\
& + C_4 \sin(b-\omega)t + C_5 \cos(b-\omega)t \\
y(t) = & D_1 + D_2 \sin(b+\omega)t + D_3 \cos(b+\omega)t \\
& + D_4 \sin(b-\omega)t + D_5 \cos(b-\omega)t,
\end{aligned}$$

where,

$$C_1 = \frac{a_1}{b^2 - \omega^2}$$

$$C_2 = \frac{(b+\omega)\dot{x}_0 + (b^2 - \omega^2)y_0 - a_2}{2b(b+\omega)}$$

$$C_3 = \frac{(b^2 - \omega^2)x_0 - (b+\omega)\dot{y}_0 - a_1}{2b(b+\omega)}$$

$$C_4 = \frac{(b-\omega)\dot{x}_0 - (b^2 - \omega^2)y_0 + a_2}{2b(b-\omega)}$$

$$C_5 = \frac{(b^2 - \omega^2)x_0 + (b-\omega)\dot{y}_0 - a_1}{2b(b-\omega)}$$

$$D_1 = \frac{a_2}{(b^2 - \omega^2)}$$

$$D_2 = \frac{(b+\omega)\dot{y}_0 - (b^2 - \omega^2)x_0 + a_1}{2b(b+\omega)}$$

$$D_3 = \frac{(b^2 - \omega^2)y_0 + (b+\omega)\dot{x}_0 - a_2}{2b(b+\omega)}$$

$$D_4 = \frac{(b^2 - \omega^2)x_0 + (b-\omega)\dot{y}_0 - a_1}{2b(b-\omega)}$$

$$D_5 = \frac{(b^2 - \omega^2)y_0 - (b-\omega)\dot{x}_0 - a_2}{2b(b-\omega)} .$$

If $b^2 = \omega^2$, then $A_{-1} = B_{-1} = 0$, and,

$$x(t) = A_3 \cos 2\omega t + A_2 \frac{\sin 2\omega t}{2\omega}$$

$$+ A_1 \frac{1 - \cos 2\omega t}{4\omega^2} + A_0 \frac{2\omega t - \sin 2\omega t}{8\omega^3}$$

$$y(t) = B_3 \cos 2\omega t + B_2 \frac{\sin 2\omega t}{2\omega}$$

$$+B_1 \frac{1 - \cos 2\omega t}{4\omega^2} + \frac{2\omega t - \sin 2\omega t}{8\omega^3} .$$

By collecting terms and using the definitions of the coefficients A_1 and B_1 , the last two equations may be written,

$$x(t) = L_1 + L_2 t + L_3 \sin 2\omega t + L_4 \cos 2\omega t$$

$$y(t) = M_1 + M_2 t + M_3 \sin 2\omega t + M_4 \cos 2\omega t,$$

where,

$$L_1 = \frac{a_1 + 2\omega \dot{y}_0 + 4\omega^2 x_0}{4\omega^2}$$

$$L_2 = \frac{a_2}{2\omega}$$

$$L_3 = \frac{2\omega \dot{x}_0 - a_2}{4\omega^2}$$

$$L_4 = - \frac{a_1 + 2\omega \dot{y}_0}{4\omega^2}$$

$$M_1 = \frac{a_2 - 2\omega \dot{x}_0 + 4\omega^2 y_0}{4\omega^2}$$

$$M_2 = - \frac{a_1}{2\omega}$$

$$M_3 = \frac{2\omega \dot{y}_0 + a_1}{4\omega^2}$$

$$M_4 = - \frac{a_2 - 2\omega \dot{x}_0}{4\omega^2} .$$